SIDE15

15TH CONFERENCE ON
SYMMETRIES AND INTEGRABILITY
OF DIFFERENCE EQUATIONS

BOOK OF ABSTRACTS AND PROGRAM

Palazzo dei Congressi, Sirmione (BS)
16–20 June 2025

Preface

It is our great pleasure to host the fifteenth edition of the international conference on Symmetries and Integrability of Difference Equations (SIDE15) at the Palazzo dei Congressi in Sirmione, Italy, from June 16th to 20th, 2025. This biennial gathering has established itself as a premier forum for sharing new developments and insights in the field of difference equations, with a particular focus on their symmetries, integrability properties, and related topics such as discrete Painlevé equations and orthogonal polynomials.

The SIDE15 conference features an impressive program of 52 contributed talks and 22 posters, showcasing cutting-edge research by experts in the field. Participants are expected to engage in stimulating discussions, foster collaborations, and exchange ideas on various aspects of difference equations.

We are delighted that our community continues to thrive globally, with scholars from diverse backgrounds contributing their expertise to advance our understanding of these fundamental mathematical structures. This Book of Abstracts presents a snapshot of the exciting research presented at SIDE15, highlighting some of the fascinating discoveries and innovations that have emerged during the meeting. We hope it will serve as a valuable resource for researchers and students alike, inspiring further exploration into the rich landscape of the regularity properties of difference equations. We also hope it will serve as a memento, capturing the state of the art on this topic in the mid-2020s.

It is worth noting that this conference marks an important milestone in the history of the SIDE series: after nearly three decades since its last appearance on Italian soil, SIDE returns to Italy with SIDE15. Indeed, the only previous SIDE conference ever held in Italy dates back to 1998, when Professor Decio Levi and Professor Orlando Ragnisco organized the SIDE3 conference in Sabaudia. We are also delighted to have Professor Ragnisco attending this event, allowing us to pay tribute to the organizers of that previous meeting, who set a high standard for future events in this beautiful country.

Sadly, we also gather for a moment of remembrance: Professor Masatoshi Noumi, who passed away in November 2024, leaves behind an invaluable legacy in the field. We will honor his memory with a special session that will bring together friends, colleagues, former students, and postdocs to share

their tributes and anecdotes about his remarkable life and work. Although not a member of SIDE's Steering Committee, Professor Noumi had been actively participating in our community for many years, attending numerous SIDE meetings and contributing significantly to the field of discrete integrable systems through his research. His most notable contribution is perhaps the outstanding review paper on the geometric theory of Painlevé equations, coauthored with Kenji Kajiwara (a member of our Steering Committee) and Yasuhiko Yamada. This work has had a profound impact on the development of the area, for instance by setting what are nowadays regarded as the standard forms of the discrete Painlevé equations.

We would like to extend our heartfelt thanks to the Steering Committee of SIDE for their tireless efforts in shaping the conference program and ensuring its smooth execution. Special recognition goes to Anton Dzhamay, Rei Inoue, Yang Shi, and Ralph Willox for their comments and suggestions, which helped to define the final form of the program you can find in this Book of Abstracts. Last but not least, we would like to thank the Chairman of the Steering Committee, Frank W. Nijhoff, for his overall supervision of all aspects of the conference. Special gratitude is also due to the funding agencies: Università degli Studi di Milano, Università degli Studi di Brescia, Università degli Studi di Bologna, Istituto Nazionale di Fisica Nucleare (INFN), and Seminario Matematico di Brescia for their generous support of our event. We also thank the Municipality of Sirmione whose help will make SIDE15 a truly memorable experience, showcasing the beauty of Sirmione as well as the richness of our field. We look forward to continuing this tradition of scientific exchange and cooperation in future editions of SIDE.

SIDE15 Organising Committee
Simonetta Abenda
Sandra Carillo
Susanna Dehò
Pavel Drozdov
Giorgio Gubbiotti (Chairman)
Danilo Latini
Pierandrea Vergallo
Federico Zullo

Contents

Talks

In the description of the talks the $*$ marks the presenting author.

1. Simonetta Abenda

Title: KP Solitons: Tropical Curves meet Grassmannians

Authors: Simonetta Abenda[1,*], Türkü Özlüm Çelik[2] Claudia Fevola[3], Yelena Mandelstham[4]

Institutions:

(1) University of Bologna and INFN, Sezione di Bologna, Italy;
(2) Max Planck Institute of Molecular Cell Biology and Genetics, Germany;
(3) Centre Inria de Saclay, France;
(4) Institute for Advanced Studies, Princeton, US.

Abstract: The object of this talk, based on [1], is to present the deep relations between KP solitons and tropical curves, in particular to show how to a given tropical curve one may associate all possible (real and regular) KP solitons compatible with the combinatorics of the metric graph representing the tropical curve. In the talk, we shall compare these new results to the various known approaches to the direct and inverse spectral problems for KP solitons [2–8].

References:

[1] S. Abenda, T. Ö, Çelik, C. Fevola, Y. Mandelshtam, KP solitons: tropical curves meet Grassmannians, in prep. 2025
[2] S. Abenda, and P. G. Grinevich, Rational degenerations of M-curves, totally positive Grassmannians and KP–solitons. *Commun. Math. Phys.* **361**(3): 1029–1081, 2018.
[3] S. Abenda, and P. G. Grinevich, Reducible M-curves for Le-networks in the totally-nonnegative Grassmannian and KP–II multiline solitons. *Sel. Math. New Ser.* **25**(3), Art. no. 43, 64 pp.,2019.
[4] S. Abenda, and P. G. Grinevich, Real regular KP divisors on M–curves and totally non-negative Grassmannians. *Lett. Mathem. Phys.* **112**(6), Paper No. 115, 64 pp., 2022.
[5] D. Agostini, T. Ö, Çelik, J. Struwe, and B. Sturmfels Theta surfaces *Vietnam J. Math.* **49**(2), 319347, 2021.

[6] D. Agostini, C. Fevola, Y. Mandelshtam, and B. Sturmfels, KP Solitons from Tropical Limits. *J. Symb. Comp.* **114**, 282–301, 2023.
[7] T. Ichikawa Periods of tropical curves and associated KP solutions. *Comm. Math. Phys.* **402**, no. 2, 1707–1723, 2023.
[8] A. Nakayashiki, On Reducible Degeneration of Hyperelliptic Curves and Soliton Solutions. *SIGMA*, **15**, Art. no. 009, 18 pp., 2019.

2. Nouf Alrubea

Title: Symmetry analysis of the geodesic equations of the canonical connection on an n-dimensional Lie algebra with a co-dimension 2 abelian nilradical

Authors: Nouf Alrubea[1,*], Ryad Ghanam [2]

Institutions:

(1) Department of Mathematics, College of Sciences, Jouf University, Saudi Arabia;
(2) Department of Liberal Arts and Sciences, Virginia Commonwealth University in Qatar, Qatar.

Type of contribution: Talk.

Abstract: In this talk we will present our preliminary results about analyzing the symmetry Lie algebra for the canonical connection on Lie groups. We will consider a general Lie algebra of dimension n that has a co-dimension two abelian nilradical. We will present examples for lower dimensional cases in dimensions four, five and six.

References:

[1] R. Ghanam, G. Thompson, E. J. Miller, Variationality of four-dimensional Lie group connections, *J. Lie Theory*, **14** (2004), 395–425.
[2] R. Ghanam, G. Thompson, Symmetry algebras for the canonical Lie group geodesic equations in dimension three, *Math. Aeterna*, **8** (2018), 37–47.
[3] Ghanam, R and Thompson, G. Lie symmetries of the canonical geodesic equations for four-dimensional Lie groups. Mathematica Aeterna, Vol. 8, 2018, no. 2, 57-70.
[4] J. Patera, R. T. Sharp, P. Winternitz, H. Zassenhaus, Invariants of real low dimension Lie algebras, *J. Math. Phys.*, **17** (1976), 986–994.
[5] R. Ghanam, G. Thompson, Lie symmetries of the canonical geodesic equations for six-dimensional nilpotent Lie groups, *Cogent Math. Stat.*, **7**, (2020), 1781505.
[6] H. Almusawa, R. Ghanam, G. Thompson, Symmetries of the canonical geodesic equations of five-dimensional nilpotent Lie algebras, *J. Generalized Lie Theory Appl.*, **13** (294), 1–5.

[7] Nouf Almutiben, Ryad Ghanam, G. Thompson and Edward L. Boone, Symmetry analysis of the canonical connection on Lie groups: six-dimensional case with abelian nilradical and one-dimensional center. AIMS Mathematics, 9(6): 1450414524.

3. Harry Braden

Title: Monopoles and Affine Toda Equations

Authors: Harry Braden[1,*], Linden Disney-Hogg[2]

Institutions:

(1) School of Mathematics and Maxwell Institute, University of Edinburgh, UK;
(2) School of Mathematics, University of Leeds, UK.

Abstract: I aim to survey connections between some affine Toda equations and a class of monopoles, the $\mathfrak{su}(2)$ BPS monopoles on R^3. Associated to such monopoles is a (finite dimensional) integrable system and, although such monopoles have been studied for nearly 50 years, the number of analytic solutions known are very few. The problem, common to many real life applications of integrable systems, is that the spectral curves have constraints placed upon them and we dont yet know how to solve for these. By imposing spatial symmetries we may reduce the equations to simpler form. Here reductions to affine Toda equations will be considered. Even in this well-studied context new and interesting questions arise. I hope the open ones will attract the experts attention. A number of new monopole solutions will be shown.

4. Sandra Carillo

Title: Fifth order soliton equations: old & new results via Bäcklund tranformations

Authors: Sandra Carillo

Institutions: SBAI Department, University of Rome "La Sapienza", Italy, & Istituto Nazionale di Fisica Nucleare, Italy.

Abstract: Fifth order nonlinear evolution equations are considered: they all are connected via Bäcklund transformations. The links, portrayed in a wide *Bäcklund Chart* [1] are here revisited. Specifically, the Caudrey-Dodd-Gibbon-Sawada-Kotera (CGDSK) and the Kaup-Kupershmidt equations

$$u_t = \left(u_{xxxx} + \frac{5}{2} u u_x + \frac{5}{12} u^3 \right)_x \ , \ q_t = \left(q_{xxxx} + 10 q q_{xx} + \frac{15}{2} q_x^2 + \frac{20}{3} q^3 \right)_x$$

where the unknowns are denoted, respectively, as u and q. They belong to the Schwartz space of smooth *rapidly decreasing functions*. The two equations are connected via Bäcklund transformations: the established links are reconsidered and new results are presented. Notable analogies with results in the

case of third order nonlinear evolution equations of *KdV type* are discussed. In addition, the two Bäcklund Chart which connect, in turn, fifth and third order soliton equations are compared.

Third order soliton equations admit non-Abelian counterparts: they are all connected via a non-Abelian Bäcklund Chart [2] Possible extensions to non-Abelian counter parts are currently under investigation [4].

References:

[1] C. Rogers, S. Carillo, *On reciprocal properties of the Caudrey-Dodd-Gibbon and Kaup-Kupershmidt hierarchies*, Physica Scripta, **36**, 865–869, (1987)
[2] S. Carillo, M. Lo Schiavo, C. Schiebold, *Abelian versus non-Abelian Bäcklund Charts: some remarks*, Evolution Equations and Control Theory, **8** (1), 43–55, (2019).
[3] S. Carillo, C. Schiebold, F. Zullo, preprint
[4] S. Carillo, C. Schiebold, *non-Abelian fifth order soliton equations*, in progress.

5. Matteo Casati

Title: Multi-component Hamiltonian difference operators

Authors: Matteo Casati[1,*], Daniele Valeri[2]

Institutions:

(1) School of Mathematics and Statistics, Ningbo University, Peoples Republic of China;
(2) Dipartimento di Matematica "Guido Castelnuovo", Università di Roma "La Sapienza", Italy, & Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy.

Abstract: We present our recent results on the classification of Hamiltonian difference operators. It is well-known that many integrable differential-difference systems possess bi-Hamiltonian structure in terms of difference operators [4]; operators of this form for scalar system have been classified up to order $(-5, 5)$ within the framework of multiplicative Poisson vertex algebras [2]. On the other hand, many years ago Dubrovin unveiled the relation between nondegenerate operators of order $(-1, 1)$ – that can be seen as discretization of Poisson brackets of hydrodynamic type – and the theory of Poisson-Lie groups [3]. We extend the classification of this latter class of operators to include degenerate cases and study the Poisson cohomology of the (degenerate) Hamiltonian structure of the Toda system. This talk is based on [1].

References:

[1] M. Casati, D. Valeri. Multi-component Hamiltonian difference operators. Preprint arXiv:2412.11772, 2024. *Under review*

[2] A. De Sole, V. G. Kac, D. Valeri, M. Wakimoto. Poisson λ Brackets for Differential-Difference Equations. *IMRN*, 13, 2020.
[3] B. A. Dubrovin. Differential-geometric Poisson brackets on a lattice. *Functional Anal. Appl.* 23(2) (1989), 57-59.
[4] F. Khanizadeh, A. V. Mikhailov, J. P. Wang. Darboux transformations and recursion operators for differential–difference equations. *Theoretical and Mathematical Physics*, 177(3):1606–1654, 2013.

6. Xiangke Chang

Title: On peakon flows of Novikov-type equations

Authors: Xiangke Chang

Institutions: LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, People's Republic of China.

Abstract: A family of integrable PDEs admit the so-called peakon solutions, the dynamics of which may be characterized by implementing the related forward/inverse spectral analysis. The spectral problems often involve non-self-adjoint operators, while the inverse spectral analysis usually involve Hermite-Padé approximation problems, which have motivated new (bi)orthogonality and random matrix models, etc. This talk is to focus on the peakon flows of the Novikov equation with a non-self-adjoint 3×3 Lax operator and one of its two-component generalizations with a non-self-adjoint 4×4 Lax operator.

7. Davide Dal Martello

Title: Painlevé VI, symmetries, and clusters

Authors: Davide Dal Martello

Institutions: Dipartimento di Matematica, Università degli studi di Padova, Padova, Italy

Abstract: The sixth Painlevé equation (PVI) has a native $\mathfrak{sl}_2(\mathbb{C})$-Fuchsian isomonodromy representation. Taking advantage of a higher Teichmüller coordinatization [2] for the corresponding monodromy group, we give Okamotos symmetry of PVI a realization [1] on the representation space in the language of cluster $\mathcal{X}$-mutations. In particular, the explicit mutation formula admits dual characterizations in geometric terms of colored associahedra and star-shaped fat graphs, expanding the cluster state of the art for PVI.

References:

[1] D. Dal Martello, *Okamoto's symmetry on the representation space of the sixth Painlevé equation.* arXiv:2411.17397, 2024.
[2] D. Dal Martello and M. Mazzocco, *Generalized double affine Hecke algebras, their representations, and higher Teichmüller theory.* Advances in Mathematics 450, 2024.

8. Adam Doliwa

Title: Quantum walks and the Ablowitz-Ladik spectral problem

Authors: Adam Doliwa

Institutions: Faculty of Mathematics and Computer Science, University of Warmia and Mazury in Olsztyn

Abstract: We define quantization scheme for discrete-time random walks on the half-line [4] consistent with Szegedy's quantization [8] of finite Markov chains. Motivated by the Karlin and McGregor description [5] of discrete-time random walks in terms of polynomials orthogonal with respect to a measure with support in the segment $[-1, 1]$, we represent the unitary evolution operator of the quantum walk [6] in terms of orthogonal polynomials on the unit circle.

We find the relation between transition probabilities of the discrete-time random walk with the Verblunsky coefficients of the corresponding polynomials of the quantum walk. We show that the both polynomials systems and their measures are connected by the classical Szegő map [7]. Our scheme can be applied to arbitrary Karlin and McGregor random walks and generalizes the so called Cantero–Grünbaum–Moral–Velázquez method [2].

It is well known [7] that polynomials orthogonal on the unit circle satisfy a recursion relation (the Szegő recurrence) of the Ablowitz–Ladik type [1]. This links quantum walks to the integrability theory in the same way as the classical random walks correspond to the Toda lattice equations [9] via the celebrated three-term relation. By our previous work [3] the Szegő recurrence allows therefore to connect quantum walks to the geometry of discrete curves.

References:

[1] M. J. Ablowitz, J. F. Ladik, Nonlinear differential-difference equations, *J. Math. Phys.*. **16**: 598–603, 1975.

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[4] A. Doliwa, A. Siemaszko, Spectral quantization of discrete random walks on half-line, and orthogonal polynomials on the unit circle. *Quantum Inf. Process.*. **23**: 384 (37 pp.), 2024.

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pp. 32–41, 2004.
[9] M. Toda, *Theory of Nonlinear Lattices*, Springer, Berlin, 1989.

9. Anton Dzhamay

Title: Discrete Painlevé Equations on the $D_5^{(1)}$ Sakai surface

Authors: Anton Dzhamay

Institutions: Beijing Institute of Mathematical Sciences and Applications (BIMSA), People's Republic of China

Abstract: The celebrated Sakai classification scheme for discrete Painlevé equations describes 22 families of rational algebraic surfaces, now called the Sakai surfaces, on which discrete Painlevé dynamics occurs. Sakai also computed the symmetries of such surface families in terms of some extended affine Weyl groups whose translation elements generate the dynamics. However, for each such surface family, there are (infinitely) many different discrete Painlevé equations. Some can be represented as compositions of simpler dynamics corresponding to short translations in the weight lattice of the corresponding affine Weyl group, but some are more interesting – these are the dynamics that lie on proper sub-families of the generic family and the corresponding symmetry group of the sub-family also generates the dynamics, just as in the generic case. Foe example, there are families of surfaces that have special curves, called the nodal curves, and the dynamics has to preserve them. Another class of examples, known as projective reductions, happens when there is a quasi-translation element that becomes a translation when restricted to special parameter values. In this talk we shall discuss a number of such examples for a particularly rich $D_5^{(1)}$ family of Sakai surfaces. Many examples are related to the equations appearing in the study of orthogonal polynomials with semi-classical weights and therefore, it may be important, especially for applications, to refine the Sakai classification scheme to include such special cases. This talk is based on a number of joint projects with Galina Filipuk, Michele Graffeo, Giorgio Gubiotti, Yang Shi, Alexander Stokes, and Ralph Willox.

10. Bao-Feng Feng

Title: Integrable discrete massive Thirring model

Authors: Junchao Chen[1], Bao-Feng Feng[2,*]

Institutions:

(1) Department of Mathematics and Institute of Nonlinear Analysis, Lishui University, Lishui, People's Republic of China;
(2) School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, USA.

Abstract: We propose integrable semi-discrete and fully discrete analogues of the massive Thirring (MT) model. Based on the bilinear structure of the MT model and its connection to the two-component Kadomtsev-Petviashvili (KP) hierarchy [1], we propose both the semi-discrete and fully discrete analogues of the MT model, along with its N-bright soliton solutions. In the continuous limit, both the discrete models and the solutions converges to the ones of the original MT model.

References:

[1] J. Chen, and B.-F. Feng, Tau-function formulation for bright, dark soliton and breather solutions to the massive Thirring model, , 150 (2023) 35-68. *Stud. Appl. Math.*. **150**: 35–68, 2023.

11. Galina Filipuk

Title: The Painlevé equivalence problem for a constrained 3D system

Authors: Galina Filipuk[1,*], Michele Graffeo[2], Giorgio Gubbiotti[3], and Alexander Stokes[1,4]

Institutions:

(1) Institute of Mathematics, University of Warsaw, Warsaw, Poland;
(2) Scuola Internazionali Studi Superiori Avanzati, Trieste, Italy;
(3) Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano, Milano, Italy;
(4) Waseda Institute for Advanced Study, Waseda University, Tokyo, Japan.

Abstract: We propose a geometric approach to study Painlevé equations appearing as constrained systems of three first-order ordinary differential equations. We illustrate this approach on a system of three first-order differential equations arising in the theory of semi-classical orthogonal polynomials. We show that it can be restricted to a system of two first-order differential equations in two different ways on an invariant hypersurface. We build the space of initial conditions for each of these restricted systems and verify that they exhibit the Painlevé property from a geometric perspective. Utilising the Painlevé identification algorithm we also relate this system to the Painlevé VI equation and we build its global Hamiltonian structure. Finally, we prove that the autonomous limit of the original system is Liouville integrable, and the level curves of its first integrals are elliptic curves, which leads us to conjecture that the 3D system itself also possesses the Painlevé property without the need to restrict it to the invariant hypersurface. In this talk we shall discuss several aspects of results in [1].

References:

[1] G. Filipuk, M. Graffeo, G. Gubbiotti, and A. Stokes, *The Painlevé equivalence problem for a constrained 3D system*, https://doi.org/10.48550/arXiv.2411.01657.

12. Vladimir S. Gerdjikov

Title: Generalized Fourier transforms for the hierarchies of MKdV equations related to $D_4^{(1)}$, $D_4^{(2)}$ and $D_4^{(3)}$ Kac-Moody algebras

Authors: Vladimir S. Gerdjikov

Institutions: Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences

Abstract: The results reported here summarize and extend the results of [1–3]. In [1] using Mikhailov reduction group [4] we constructed the three nonequivalent gradings of the algebra $D_4 \simeq so(8)$. The first one is the standard one obtained with the Coxeter automorphism $C_1 = S_{\alpha_2} S_{\alpha_1} S_{\alpha_3} S_{\alpha_4}$ using its dihedral realization. Here and below α_j, $j = 1, \dots, 4$ are the simple roots of D_4 algebra, and S_{α_j} are the Weyl reflections corresponding to α_j. In the second one we use $C_2 = S_{\alpha_1} S_{\alpha_3} S_{\alpha_2} R$ where R is the mirror automorphism of D_4. The third one is $C_3 = S_{\alpha_2} S_{\alpha_1} T$ where T is the external automorphism of order 3. For each of these gradings we constructed the basis in the corresponding linear subspaces $\mathfrak{g}^{(k)}$, the orbits of the Coxeter automorphisms and the related Lax pairs generating the corresponding mKdV hierarchies. We found compact expressions for each of the hierarchies in terms of the recursion operators. At the end we wrote explicitly the first nontrivial mKdV equations and their Hamiltonians. For $D_4^{(1)}$ these are in fact two mKdV systems, due to the fact that in this case the exponent 3 has multiplicity 2. Each of these mKdV systems consist of 4 equations of third order with respect to ∂_x. For $D_4^{(2)}$ this is a system of three equations of third order with respect to ∂_x. Finally, for $D_4^{(3)}$ this is a system of two equations of fifth order with respect to ∂_x.

In [2, 3] we extend the results of [5] to NLEE, related to Lax pairs with $\mathbb{Z}_h$ Mikhailov reduction groups. Using the contour integration method, applied to properly chosen Green function we derived the completeness relations for the corresponding 'squared solutions' of the Lax operator. Thus the 'squared solutions' may be viewed as generalized exponentials linearizing each of the hierarchies of NLEE. Effectively they also provide the spectral decompositions of the recursion operators mentioned above. Besides we propose a generalization of Zakharov-Shabat [6] dressing method that allows one to derived the soliton solutions of the above equations.

References:

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[2] V S Gerdjikov, A B Yanovski. CBC systems with Mikhailov reductions by Coxeter Automorphism. I. Spectral Theory of the Recursion Operators. Studies in Applied Mathematics **134** (2), 145–180 (2015). DOI: 10.1111/sapm.12065.
[3] V. S. Gerdjikov, A B Yanovski. Riemann-Hilbert Problems, families of commuting operators and soliton equations Journal of Physics: Conference Series **482** (2014) 012017 doi:10.1088/1742-6596/482/1/012017
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[5] V.S. Gerdjikov, Generalised Fourier transforms for the soliton equations. Gauge covariant formulation Inverse Problems 2, 51-74 (1986).
[6] V.E. Zakharov and A.B. Shabat, Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II., Funct. Anal. Appl. 13, 166-174 (1979).

13. Claire R. Gilson

Title: Frieze Patterns, Wronskians and Discrete integrable systems.

Authors: Claire R. Gilson

Institutions: School of Mathematics & Statistics, University of Glasgow, UK.

Abstract: Frieze patterns were developed by Coxeter in the 1970's [1]. They are patterns of numbers that can be generated from simple rules involving determinants. In recent years they have come back into view due to their connections with cluster algebras in the work of Fomin and Zelevinsky [2]. In this talk I will review the definition of a frieze pattern and examine their connections with determinants, identities and discrete integrable systems.

References:

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[2] S. Fomin, and A. Zelevinsky, Cluster algebras I: foundations, *J. Am. Math. Soc.*, **15**, 497–529, 2002.

14. Jarmo Hietarinta

Title: Solutions to the constant Yang-Baxter equation: additive charge conservation in three dimensions

Authors: Jarmo Hietarinta[1,*], Paul Martin[2], Eric C. Rowell[2]

Institutions:

(1) Department of Physics and Astronomy, University of Turku, Finland;
(2) School of Mathematics, University of Leeds, UK.
(3) Texas A&M University, USA.

Abstract: The Yang-Baxter equation is difficult to solve even in the constant form $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ and a complete solution is known only for rank two. For further progress it is important to make a meaningful ansatz. Recently Martin and Rowell proposed charge-conservation as an effective constraint [1]. We explore the results obtained in a three-state case by a slightly different charge-conservation rule [2].

References:

[1] P. Martin, E. C. Rowel: *Classification of spin-chain braid representations*, `arXiv:2112.04533v3`.

[2] J. Hietarinta., P. Martin and E. C. Rowell: *Solutions to the constant YangBaxter equation: additive charge conservation in three dimensions*, Proc. R. Soc. A **480,** 20230810 (24pp) (2024) doi:10.1098/rspa.2023.0810 or 10.48550/arXiv.2310.03816

15. Xing-Biao Hu

Title: A Discrete Integrable System Associated with Bivariate Orthogonal Polynomials on Genus-Two Hyper-Elliptic Curves

Authors: Jing-Rui Wu and Xing-Biao Hu*

Institutions: LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, People's Republic of China & School of Mathematical Sciences, University of Chinese Academy of Sciences, People's Republic of China

Type of contribution: Talk.

Abstract: We present a new discrete integrable system derived from the recurrence structure of bivariate orthogonal polynomials associated with genus-two hyper-elliptic curves. By expressing these polynomials through determinantal formulas based on a structured sequence of monomials constrained by the curve equation, we obtain linear recurrence relations and construct two distinct Lax pairs. The compatibility condition of these pairs leads to a nonlinear system, extending earlier integrable models such as the higher analogue of the discrete-time Toda (HADT) equation and the quotient-quotientdifference (QQD) scheme. Furthermore, we derive a coupled multilinear relation formulated entirely in terms of determinants defined by the underlying moment functionals. This work illustrates how algebraic curves of higher genus enrich the theory of discrete integrable systems.

16. Pavlos Kassotakis

Title: On quadrirational pentagon maps

Authors: Pavlos Kassotakis

Institutions: Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw,

Abstract: In this talk, we present equivalence classes of rational solutions of a specific type of the set theoretical version of the pentagon equation. That is, up to Möbius transformations, we find quadrirational one-component maps of rational functions in two arguments that serve as solutions of the pentagon equation. Also, provided that a pentagon map admits at least one partial inverse, we obtain entwining pentagon maps that allow us to construct YangBaxter maps.

17. Thomas Kecker

Title: Geometric approach for quasi-Painlevé Hamiltonian systems

Authors: Thomas Kecker [1,*], Marta Dell'Atti [2]

Institutions:

(1) University of Portsmouth, UK,
(2) University of Warsaw, Poland.

Type of contribution: Talk.

Abstract: We study second-order differential equations and Hamiltonian systems of quasi-Painlevé type by a geometric approach, originally applied to the Painlevé equations by K. Okamoto to obtain their spaces of initial conditions. Quasi-Painlevé equations, first so named by S. Shimomura, possess the property that all movable singularities of their solutions in the complex plane are algebraic poles. To obtain their spaces of initial conditions, a larger number of blow-ups is required than in the Painlevé case. We give examples of quasi-Painlevé equations and systems and show how they can be related through bi-rational transformations, by comparing the irreducible components of their exceptional divisors arising in the blow-up process. This presentation is mainly based on the two articles [1] and [2], published jointly with M. Dell'Atti.

References:

[1] M. Dell'Atti and T. Kecker, Geometric approach for the identification of Hamiltonian systems of quasi-Painlevé type, *J. Phys. A: Math. Theor.* **58**: 095202, 2025, (doi: 10.1088/1751-8121/adb819)
[2] M. Dell'Atti and T. Kecker, Spaces of initial conditions for quartic Hamiltonian systems of Painlevé and quasi-Painlevé type, preprint, *arXiv: 2412.17135*, 2025

18. Wookyung Kim

Title: Discrete integrable systems associated with the deformation of cluster maps of type D

Authors: Wookyung Kim

Institutions: Graduate School of Mathematical Sciences, the University of Tokyo.

Abstract: A discrete integrable system in cluster algebra can be described by the iteration of the Liouville-integrable cluster map, which consists of cluster mutations. One can construct a family of such discrete dynamical systems by applying a novel approach: the deformation of cluster mutations introduced by Hone and Kouloukas in [1]. This approach modifies the cluster mutations in a way that preserves the natural presymplectic form. Recently, in [2, 3], we have identified several successful examples of deformed integrable maps that retain integrability but no longer exhibit the Laurent property.

In this talk, we will present a deformation of an integrable cluster map associated with Dynkin type D_4. We will then demonstrate how the deformed map restores the Laurent property using a method known as Laurentification, which lifts the map into a higher-dimensional space where the Laurent property holds. Furthermore, we will show that this Laurentification enables us to find a new cluster algebra of higher rank equipped with a unique quiver. Finally, we will illustrate the generalization of the D_4 case to the deformation of cluster maps of type D_{2N}, which can also be lifted via Laurentification to new cluster maps associated with a special family of quivers.

References:

[1] A. N. W. Hone, and T. Kouloukas, Deformation of cluster mutation which preserves presymplectic form, *J Algebr Comb* **57**, 763-791, 2023.

[2] A. N. W. Hone, W. Kim, and T. Mase, New cluster algebras from old: integrability beyond Zamolodchikov periodicity, *JPhys. A: Math. Theor.* **57** 415201, 2024.

[3] J. E. Grabowski, A. N. W. Hone, and W. Kim, Deformed cluster map of type A_{2N}, preprint, arXiv:2402.18310, 2024.

19. Roman Kozlov

Title: Symmetries and first integrals of variational ODEs with delay

Authors: Roman Kozlov

Institutions: Department of Business and Management Science, Norwegian School of Economics, Helleveien 30, 5045, Bergen, Norway

Abstract: A Lagrangian formalism for variational second-order delay ordinary differential equations (DODEs) is developed. The Noether operator identity for a DODE is established, which relates the invariance of a Lagrangian function with the appropriate variational equations and the conserved quantities. The identity is used to formulate Noether-type theorems that give the first integrals for DODE with symmetries. Relations between the invariance of the variational second-order DODEs and the invariance of the Lagrangian functions are also analyzed. Several examples illustrate the theoretical results.

References:

[1] V. Dorodnitsyn, R. Kozlov, and S. Meleshko, Lagrangian formalism and Noether-type theorems for second-order delay ordinary differential equations, *J. Phys. A: Math. Theor.* **56** (34) 345203, 2023.

20. Tomas Lasic Latimer

Title: Orthogonal polynomials, q-difference equations and discrete Painlevé.

Authors: Tomas Lasic Latimer

Institutions: Mathematics department, University of California Santa Cruz, USA.

Abstract: I discuss recent work [1, 2] on finding an accurate asymptotic approximation of orthogonal polynomials whose measure is supported on the discrete q-lattice. In searching for these asymptotic results, deep connections between q-difference calculus, the Riemann-Hilbert problem and discrete Painlevé equations are unearthed. We will touch on some of these connections and how they show themselves in the context of orthogonal polynomials.

References:

[1] N. Joshi, and T. Lasic Latimer, Asymptotics of discrete q-Freud II orthogonal polynomials from the q-Riemann Hilbert problem. *Nonlinearity.* **36** (8): 3969, 2023.

[2] T. Lasic Latimer, Asymptotics for multiple q-orthogonal polynomials from the Riemann-Hilbert Problem. *arXiv:2502.00335.* 2025.

21. Irfan Mahmood

,

Title: Noncommutative Painlevé second equation and its associated Toda chains

Authors: Irfan Mahmood

Institutions: Centre for High Energy Physics, University of the Punjab, Pakistan

Abstract: The extension of the Painlevé equations to noncommutative spaces has been extensively investigated in the theory of integrable systems. An interesting topic is the exploration of some remarkable aspects of these equations, such as the Painlevé property, the Lax representation and the Darboux transformation, and their connection to well-known integrable equations. This discussion addresses the Lax formulation, the Darboux transformation and a quasideterminant solution of the noncommutative analog of Painlevé second equation. This also encloses quasideterminant solutions to the noncommutative Painlevé second equation by taking the Toda solutions at $n = 1$ as a seed solution in its Darboux transformations and derivation

of Lax pair of associated system of non-linear differential equations of Toda system with $n = 1$.

22. Ievgen Makedonskyi

Title: Duality theorems for staircase matrices

Authors: Ievgen Makedonskyi[1,*], Anton Khoroshkin[2], Evgeny Feigin[3]

Institutions:

(1) Beijing Institute of Mathematical Sciences and Application, (BIMSA), People's Republic of China;
(2) Department of Mathematics, University of Haifa, Israel;
(3) School of Mathematical Sciences, Tel Aviv University, Israel.

Abstract: The well known Cauchy identity expresses the product of terms $(1 - x_i y_j)^{-1}$ for (i, j) indexing entries of a rectangular $m \times n$-matrix as a sum over partitions λ of products of Schur polynomials: $s_\lambda(x)s_\lambda(y)$. Algebraically, this identity comes from the decomposition of the symmetric algebra of the space of rectangular matrices, considered as a $\mathfrak{gl}_m$-$\mathfrak{gl}_n$-bi-module. We generalize the Cauchy decomposition by replacing rectangular matrices with arbitrary staircase-shaped matrices equipped with the left and right actions of the Borel upper-triangular subalgebras. For any given staircase shape Y we describe left and right "standard" filtrations on the symmetric algebra of the space of shape Y matrices. We show that the subquotients of these filtrations are tensor products of Demazure and opposite van der Kallen modules over the Borel subalgebras. On the level of characters, we derive three distinct expansions for the product $(1 - x_i y_j)^{-1}$ for $(i, j) \in \mathsf{Y}$. The first two expansions are sums of products of key polynomials $\kappa_\lambda(x)$ and (opposite) Demazure atoms $a^\mu(y)$. The third expansion is an alternating sum of products of key polynomials $\kappa_\lambda(x)\,\kappa^\mu(y)$.

References:

[1] Evgeny Feigin, Anton Khoroshkin, Ievgen Makedonskyi, *Cauchy identities for staircase matrices*, arXiv:2411.03117.
[2] Anton Khoroshkin, Ievgen Makedonskyi *Bubble sort and Howe duality for staircase matrices*, arXiv:2502.21184.

23. Ian Marquette

Title: Superintegrable deformations of Lotka-Volterra systems : A commutant approach

Authors: Peter H. van der Kamp[1], Ian Marquette[2,*], GRW Quispel[1], Pol Vanhaecke[3]

Institutions:

(1) Department of Mathematics, La Trobe University, Australia;

(2) Department of Mathematical and Physical Sciences, La Trobe University, Australia;
(3) Université de Poitiers Laboratoire de Mathématiques et Applications, France.

Abstract: The Lie-algebraic notion of commutant was used in context of superintegrable systems such as the generic system on the sphere, algebraic superintegrable systems from Cartan subalgebra, missing label problem and subalgebra chains in nuclear physics [1–3]. In this talk we consider the Lie-algebraic notion of commutant in the setting of quadratic algebras in context of certain Lotka-volterra systems [4, 5]. This provides a framework for deforming Hamiltonian differential equations. By taking a subalgebra of the algebra of integrals, and considering the set of functions that Poisson commute with that subalgebra, the Hamiltonian can be deformed in two different ways, while retaining integrability. We deform Liouville integrable and superintegrable Lotka-Volterra systems. We present different explicit constructions considering Abelian and non-Abelian symmetry subalgebras. In previous work related to semi simple Lie and their Poisson-Lie algebras, the elements of the commutant were polynomials, in the context of this talk the element of the commutant will be rational and determined via solving systems of partial differential equations. We obtain superintegrable systems for specific dimensions, and in arbitrarily dimensions. This talk is mainly based on recent results [5]. It will be also pointed out how the construction of commutant being performed in abstract setting can accommodate different types of operators and realizations of the underlying algebraic structures.

References:

[1] F. Correa, M. A. del Olmo, I. Marquette, J. Negro, Polynomial algebras from su(3) and the generic model on the two sphere, Journal of Physics A: Mathematical and Theoretical 54 (1), 015205 (2021)
[2] R. Campoamor-Stursberg, D. Latini, I Marquette, Y.-Z. Zhang, Algebraic (super-)integrability from commutants of subalgebras in universal enveloping algebras, Journal of Physics A: Mathematical and Theoretical 56 (4) (2023)
[3] R. Campoamor-Stursberg, D. Latini, I. Marquette, J. Zhang, Y.-Z. Zhang, On the construction of polynomial Poisson algebras: a novel grading approach, arXiv:2503.03490
[4] P. H. van der Kamp, T. E. Kouloukas, G. R. W. Quispel, D. T. Tran and Pol Vanhaecke, Integrable and superintegrable systems associated with multi-sums of products, Proc. R. Soc. A 470: 20140481. (2014)
[5] P. H. van der Kamp, I. Marquette, G.R.W Quispel and P. Vanhaecke, Symmetry algebras, commutants and superintegrable deformations of Lotka-Volterra systems (in preparation)

24. Takafumi Mase

Title: Exact calculation of degrees for lattice equations: a singularity approach

Authors: Takafumi Mase

Institutions: Graduate School of Mathematical Sciences, the University of Tokyo.

Abstract: The theory of degree growth and algebraic entropy plays a crucial role in the field of discrete integrable systems. However, a general method for calculating degree growth for lattice equations (partial difference equations) is not yet known.

In this talk, I will propose a new method to rigorously compute the exact degree of each iterate for lattice equations. Halburd's method, which is a novel approach to computing the exact degree of each iterate for mappings (ordinary difference equations) from the singularity structure, forms the basis of our idea. The strategy is to extend this method to lattice equations.

First, I will illustrate, without rigorous discussion, how to calculate degrees for lattice equations using the lattice version of Halburd's method and discuss what problems we need to solve to make the method rigorous. Then, I will provide a framework to ensure that all calculations are accurate and rigorous. If time permits, I would also like to discuss how to detect the singularity structure of a lattice equation.

References:

[1] T. Mase, Exact calculation of degrees for lattice equations: a singularity approach. *preprint*, arXiv:2402.16206.

25. Ben Mitchell

Title: Special Function Solutions of the Fifth Painlevé Equation

Author: Peter A. Clarkson[1], Anton Dzhamay[2], Andrew N.W. Hone[1], and Ben Mitchell[1,*]

Institution:

(1) School of Engineering, Mathematics and Physics University of Kent, UK;
(2) Beijing Institute of Mathematical Sciences and Applications (BIMSA), People's Republic of China.

Abstract: We explore rational solutions of the fifth Painlevé equation. This equation exhibits special function solutions for specific parameter values, expressed in terms of Kummer functions [1, 2]. It is well-known that the third Painlevé equation has special function solutions represented by Bessel functions. By utilizing connection formulae between Kummer functions and modified Bessel functions, we demonstrate that the fifth Painlevé equation also admits Bessel function solutions. Furthermore, we investigate the application of these solutions to discrete Painlevé equations [3].

References:

[1] T. Masuda, Classical Transcendental Solutions of the Painlevé Equations and Their Degeneration, Tohoku Math. J., 2004, 56, **4**, 467-490.

[2] P.J. Forrester and N.S. Witte, Application of the τ-function theory of Painlevé equations to random matrices:P_V, P_{III}, the LUE, JUE, and CUE, *Comm. Pure Appl. Math.*, **55** (2002), no. 6, 679-727.

[3] P. A. Clarkson, A. Dzhamay, A.N.W. Hone and B. Mitchell. Special solutions of a discrete Painlevé equation for quantum minimal surfaces, 2025. Preprint available at https://arxiv.org/abs/2503.14436.

26. Maciej Nieszporski

,

Title: Integrable discretization of chiral models and discrete analytic functions theories.

Authors: Maciej Nieszporski

Institutions: Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw,

Abstract: Chiral models (or nonlinear sigma-models) play a prominent role in physics. I will present current state of the art in integrable discretization of these models, with special emphasis on recent achievements in this field. In particular, I will discuss how discrete chiral models relate to theories of discrete analytic functions.

27. Frank Nijhoff

Title: On the elliptic lattice Korteweg-de Vries equation: a curious discrete integrable system

Authors: Frank Nijhoff[1,2], Cheng Zhang[2], and Dajun Zhang[2]

Institutions:

(1) School of Mathematics, University of Leeds, UK;
(2) Department of Mathematics, Shanghai University, People's Republic of China.

Abstract: The elliptic lattice KdV system was introduced in [1] as a natural elliptic generalisation of the lattice potential KdV equation associated with an elliptic Cauchy kernel in the underlying direct linearisation scheme. It exhibits many of the usual integrability features: a Lax representation, multi-soliton solutions and integrable reductions. However, compared to the usual lattice potential KdV equation (H1 of the ABS list) it lacks a number of important features: the Lax matrices no longer factorise, there is no Lagrangian structure known (so far) and the simplest nontrivial periodic solutions are already of higher genus (compared to $g = 1$ for the usual case of H1). Relation to the well-known Q4 or Adler's equation of [2], or to any

other quadrilateral lattice equations, are yet to be estabished. In the present note, I will present some novel results on this elliptic lattice KdV system: a coupled set of quadrilateral multiquartic equations (in contrast to the multiquadratic equationss of [3]), and an associated elliptic Yang-Baxter map. Furthermore, associated 2×2 matrix systems subject to curious restrictions on the entries will be presented, as well an elliptic generalisation of the "hierarchy generating PDE" that arose as a continuous counterpart of the H1 equation in [6].

References:

[1] F. W. Nijhoff, and S. Puttock, On a two-parameter extension of the lattice KdV system associated with an elliptic curve. *J. Nonl. Math. Phys.*. **10**, Suppl. 1: 107–123, 2003.

[2] V. E. Adler, Bäcklund transformation for the Krichever-Novikov equation. *International Mathematics Research Notices*. **1998**, Issue 1: 1–4, 1998.

[3] J. Atkinson, and M. Nieszporski, Multi-quadratic quad equations: integrable cases from a factorized-discriminant hypothesis. *International Mathematics Research Notices*. **2014**, Issue 15: 4215–4240, 2014.

[4] P. Jennings, and F. W. Nijhoff, On an elliptic extension of the KadomtsevPetviashvili equation. *J. Phys. A: Math. Theor.* **47**, No. 5: 055205, 2014.

[5] W. Fu, and F.W. Nijhoff, On a coupled KadomtsevPetviashvili system associated with an elliptic curve. *Stud. Appl. Math.*. **149**, Issue 4: 1086–1122, 2022.

[6] F. W. Nijhoff, A. Hone, and N. Joshi, On a Schwarzian PDE associated with the KdV hierarchy. *Physics Letters A*. **267** (2-3): 147–156, 2000.

28. Yousuke Ohyama

Title: Connection problem of q-linear equations and the q-Painlevé fifth equation

Authors: Yousuke Ohyama

Institutions: Department of Mathematical Sciences, Tokushima University, Japan.

Abstract: The connection problem on the q-Painlevé VI was studied by Mano [1]. He showed that connection matrix of a linear equation corresponding the q-Painlevé VI is decomposed into a product of connection matrices of two basic hypergeometric equations. Based on Mano's decomposition, we study a character variety of the q-Painlevé VI [2]. In this presentation we study a confluent case of Mano's decomposition for the q-Painlevé fifth equation. The connection matrix is represented by a product of the connection matrix of ${}_2\phi_1$ and ${}_1\phi_1$.

References:

[1] T. Mano, Asymptotic behaviour around a boundary point of the q-Painlevé VI equation and its connection problem. *Nonlinearity.* **23** (7), 1585–1608, 2010.
[2] Y. Ohyama, Y.; J.-P. Ramis; J. Sauloy; The space of monodromy data for the Jimbo-Sakai family of q-difference equations. *Ann. Fac. Sci. Toulouse, Math.* (6) **29** (5), 1119–1250, 2021.

29. Kanam Park

Title: A symmetry of a 3×3 Lax form for the q-Painlevé equation of type $E_6^{(1)}$

Authors: Kanam Park

Institutions: National Institute of Techonology, Toba College

Abstract: The q-Painlevé equation of type $E_6^{(1)}$ was given as a q-difference system with an affine Weyl group symmetry of type $E_6^{(1)}$. In the previous work [1], we gave a new 3×3 matrix Lax form for the equation whose Lax matrix was factorized. In [2], bi-rational Weyl group actions on certain matrix Lax operators given in factorized form were studied.

Our goal is to understand a symmetry of the q-Painlevé equation of type $E_6^{(1)}$ as actions on the 3×3 matrix Lax form [1]. In this talk, we show these actions consist of bi-rational Weyl group actions considered in [2] which generate $W(A_2 + A_2 + A_2)$ and a transformation as a result of a q-middle convolution [3], [4] with a special choice of a parameter.

References:

[1] Park, K., "A 3×3 Lax form for q-Painlevé equations of type E_6", (2023), SIGMA **19**, 094, 17 pages
[2] Park, K. and Yamada, Y., "Symmetry of factorized Lax matrices", RIMS Kôkyûroku Bessatsu, **B 87**, (2021), 133–145.
[3] Sakai, H. and Yamaguchi, M., "Spectral types of linear q-difference equations and q-analogue of middle convolution", (2017), IMRN, **2017**, 1975–2013.
[4] Sasaki, T., Takagi, S. and Takemura, K., "q-middle convolution and q-Painlevé equation", (2022), SIGMA, **18**, 21pages.

30. Linyu Peng

Title: Symmetries of differential-difference equations and Noether's theorems

Authors: Linyu Peng[1,*] , Peter Hydon[2]

Institutions:

(1) Department of Mechanical Engineering, Keio University, Japan;
(2) School of Engineering, Mathematics and Physics University of Kent, UK;

Abstract: Symmetries have proven to be of fundamental importance in understanding physical models, owing to their versatile applications in elucidating the properties of their solutions. Since the 1990s, significant efforts have been made to extend symmetry methods to differential-difference equations (DDEs), such as the Toda lattice and the Volterra equation, recognising their significance as models for describing physical phenomena. Despite these advancements, several unresolved issues continue to challenge the field–for example, the problem of non-commutativity of prolonged infinitesimal symmetry generators [1, 2]. In this talk, we will address this issue and present a solution that also enables us to establish a DDE version of Noether's theorems [3].

References:

[1] D. Levi , P. Winternitz, and R. I. Yamilov, Lie point symmetries of differential-difference equations. *J. Phys. A: Math. Theor.* **43**: 292002, 2010.

[2] L. Peng, Symmetries, conservation laws, and Noether's theorem for differential-difference equations. *Stud. Appl. Math.* **139**: 457–502, 2017.

[3] L. Peng, and P. E. Hydon, Transformations, symmetries and Noether theorems for differential-difference equations. *Proc. R. Soc. A* **478**: 20210944, 2022.

31. Orlando Ragnisco

Title: The Volterra Maximally Superintegrable cases: novel analytical and numerical results

Authors: Orlando Ragnisco[1,*], Federico Zullo[2],

Institutions:

(1) Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre;
(2) DICATAM, Università degli Studi di Brescia, Italy & Istituto Nazionale di Fisica Nucleare Sezione di Milano Bicocca, Italy.

Abstract: The results presented in my talk are a natural development of those described in the paper [1], where the authors reconsidered the integrable case of the Hamiltonian N-species Lotka-Volterra system, introduced by Vito Volterra in 1937 [2]. There, an alternative approach for constructing the integrals of motion has been proposed, and compared with the old Volterra approach. Here we go beyond those results, and show that in fact the model introduced by Volterra and studied by us is not just integrable, but is maximally superintegrable and reducible to a system with only one degree of freedom regardless the number of species considered. Analytical and numerical results are also given.

References:

[1] M. Scalia, O. Ragnisco, B. Tirozzi, and F. Zullo, The Volterra Integrable case. Novel analytical and numerical results, *OCNMP* **4** (2024) pp. 188–211.

[2] V. Volterra, Principes de Biologie Mathematiques, Acta biotheoretica (Leiden, 3, 6-39), (1937).

32. John A. G. Roberts

Title: Arithmetic and geometric aspects of the (symbolic) dynamics of piecewise-linear maps

Authors: John A. G. Roberts[1], Asaki Saito[2], and Franco Vivaldi[3]

Institutions:

(1) School of Mathematics and Statistics, UNSW Sydney, Australia;
(2) Future University Hakodate, Japan;
(3) School of Mathematical Sciences, Queen Mary, University of London, UK.

Abstract: We consider a family of planar area-preserving maps, which are linear but different on each of the right and left half-planes. Such maps, studied extensively by Lagarias and Rains, can support periodic and quasiperiodic dynamics with a foliation of the plane by invariant curves. The parameter space is two dimensional and the set of parameters for which an initial condition on the half-plane boundary returns to it are algebraic "critical" curves, determined by the symbolic dynamics of the itinerary between the boundaries. An important component of the planar dynamics is the dynamics it induces on the unit circle topologically conjugate to rigid rotation – with induced symbolic dynamics and critical curves. We study arithmetic, algebraic, and geometric aspects of the planar and circle (symbolic) dynamics with connections to Farey sequences, continued fraction expansions, quasi-Sturmian irrationals and continuant polynomials. This talk is based on the papers [1, 2].

References:

[1] J. A. G. Roberts, A. Saito, and F. Vivaldi, Critical curves of a piecewise-linear map, *Chaos* **31** 073134 (19pp), 2021.
[2] J. A. G. Roberts, A. Saito, and F. Vivaldi, Critical curves of rotations, *Indag. Math.* **35** 989–1008, 2024.

33. Hidetaka Sakai

Title: Symplectic structure of discrete Painlevé equations

Author: Takafumi Mase[1], Akane Nakamura[2], and Hidetaka Sakai[1,*]

Institutions:

(1) Graduate School of Mathematical Sciences, the University of Tokyo.
(2) Department of Mathematics, Faculty of Science, Josai University, Japan.

Abstract: When considering the higher dimensional generalization of the discrete Painlevé equations, simply writing down the birational maps would result in a complicated description. As we recall from the case of differential equations, it was necessary for them to be written in terms of a Hamiltonian

in order to write the equations simply. The symplectic structure is also useful for simplifying calculations when computing compatibility conditions of linear equations. In this talk, I would like to explain an attempt to describe the discrete Painlevé equations using a symplectic structure [1]. Furthermore, we also discuss generalization to higher dimensions.

References:

[1] T. Mase, A. Nakamura, and H. Sakai: "Discrete Hamiltonians of discrete Painlevé equations", Ann. Fac. Sci. Toulouse Math. **29** (2020), no. 5, 1251–1264.

34. Wolfgang K. Schief

Title: Integrability vs consistency on"hexagonal lattices"

Authors: Wolfgang K. Schief

Institution: School of Mathematics and Statistics, The University of New South Wales, Australia.

Abstract: The notion of "consistency around the cube" of discrete equations defined on the quadrilaterals of $\mathbb{Z}^n$ lattices has turned out to be both theoretically useful and applicable in a practical sense. The study of the consistency of discrete equations which do not fall into the above category is still in its infancy. In this talk, I will present some ideas on the consistency of discrete equations which are defined on the hexagons of a honeycomb lattice. This will lead to questions about the connection between integrability and consistency. For instance, it is known that the multiratio relation of Menelaus type is multi-dimensionally consistent on lattices of A_n type and universally accepted as being integrable, while the multiratio relation of Ceva type is not. However, the latter is consistent on the type of "hexagonal lattice" discussed in this talk.

35. Anton Shchechkin

Title: Quantum Painlevé/refined gauge theory correspondence

Authors: Giulio Bonelli, Anton Shchechkin*, Alessandro Tanzini

Institutions: Scuola Superiori Studi Avanzati, Italy & Istituto Nazionale di Fisica Nucleare Sezione di Trieste, Italy, & Institute for Geometry and Physics, Italy.

Abstract: We present bilinear tau forms of quantum Painlevé equations and their solutions around critical points. Specifically, we show that these solutions are given by the Zak transform of $\mathcal{N} = 2$ $D = 4$ $SU(2)$ gauge theory partition functions in general Ω-background, following the approach of the seminal paper [1]. These partition functions appear as expansions around critical points of the (quantum) Painlevé equations, and we focus on (asymptotic) expansions around the irregular critical point $t = \infty$. Such expansions

correspond to the strong coupling regime of the gauge theories, including Argyres-Douglas points. On the gauge theory side, we compute the partition functions using refined holomorphic anomaly equations. We also compare our solutions with those expressed in terms of irregular Virasoro conformal blocks, obtained in [2, 3].

This talk is based on [4] and results in preparation.

References:

[1] O. Gamayun, N. Iorgov, and O. Lisovyy, *Conformal field theory of Painlevé VI*. In: JHEP 10, 183 (2012), p. 038. arXiv: 1207.0787 [hep-th]

[2] Hajime Nagoya, *Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations*. In: J. Math. Phys. 56.12 (2015), p. 123505. arXiv: 1505.02398 [math-ph]

[3] Hajime Nagoya, *Remarks on irregular conformal blocks and Painlevé III and II tau functions*. arXiv: 1804.04782 [math-ph]

[4] G. Bonelli, A. Shchechkin, A. Tanzini, *Refined Painlevé/gauge theory correspondence and quantum tau functions*. arXiv: 2502.01499 [hep-th]

36. Yang Shi

Title: New symmetries of discrete Painlevé equations from Normalizer theory of Coxeter groups

Authors: Yang Shi

Institutions: College of Science and Engineering, Flinders University, Australia.

Abstract: Recently, we reviewed [1] some properties of the affine Weyl group in the context of their applications to discrete integrable systems such as Sakai's list of discrete Painlevé equations. In particular, Normalizer theory of Coxeter group is used to construct some quasi-translational elements of the Weyl groups. They are found to give rise to the dynamics of various discrete integrable equations as subcases of Sakai's equations.

References:

[1] Y. Shi, Translations in affine Weyl groups and their applications in discrete integrable systems, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 481, 2312, 20240749, 2025.

37. Anup Anand Singh

Title: Lagrangian multiforms for finite-dimensional integrable hierarchies

Authors: Anup Anand Singh[1]

Institutions:

(1) School of Mathematics, University of Leeds, UK.

Abstract: First introduced in 2009 in the discrete setup, Lagrangian multiforms provide a variational framework for describing integrable hierarchies using a generalised variational principle applied to an appropriate generalisation of a classical action. In this talk, I will give an overview of this framework and report some recent results based on recent works [1–3].

In particular, I will discuss two different approaches to constructing Lagrangian multiforms for finite-dimensional integrable hierarchies. The first approach [1, 3] draws from the theory of Lie dialgebras to produce Lagrangian multiforms living on coadjoint orbits. Lie dialgebras are related to Lie bialgebras but are more flexible in that they incorporate the case of non-skew-symmetric r-matrices. The second approach [2], more geometric in flavour, is rooted in gauge theory. I will explain how hierarchies of Hitchin integrable systems arise from a certain gauge fixing of 3-dimensional topological-holomorphic BF theory, and how this connection can be exploited to provide a variational description of these hierarchies.

Finally, I will illustrate these constructions using the examples of (rational and elliptic) Gaudin models and the elliptic spin Calogero-Moser model, and time permitting, discuss some possible applications.

References:

[1] V. Caudrelier, M. Dell'Atti, A.A. Singh, *Lagrangian multiforms on coadjoint orbits for finite-dimensional integrable systems*, Lett. Math. Phys. **114** (2024), 34.

[2] V. Caudrelier, D. Harland, A.A. Singh, B. Vicedo, *3d mixed BF multiform and Hitchin integrable hierarchies*, In preparation.

[3] V. Caudrelier, A.A. Singh, B. Vicedo, *Lagrangian multiforms for cyclotomic Gaudin models*, SIGMA **20** (2024), 100.

38. Sergey Smirnov

Title: Darboux formulas for linear hyperbolic equations in discrete case

Authors: Sergey Smirnov

Institutions: School of Mathematics & Statistics, University of Glasgow, UK.

Abstract: In the second half of the 19th century Darboux obtained determinant formulas that provide the general solution for a linear hyperbolic second order PDE with finite Laplace series. These formulas played an important role in his study of the theory of surfaces and, in particular, in the theory of conjugate nets. During the last three decades discrete analogs of conjugate nets (Q-nets) were actively studied in Discrete Differential Geometry. Laplace series can be defined also for hyperbolic difference operators. We prove discrete analogs of Darboux formulas for discrete and semi-discrete hyperbolic operators with finite Laplace series.

39. Alexander Stokes

Title: Monodromy of monodromy manifolds and symmetries of Sakai surfaces

Authors: Pieter Roffelsen[1], Alexander Stokes[2,*].

Institutions:

(1) School of Mathematics and Statistics, The University of Sydney, Australia;
(2) Waseda Institute for Advanced Study, Waseda University, Tokyo, Japan.

Abstract: Painlevé equations are associated to complex algebraic surfaces in two different ways, which are related by biholomorphism under different instances of the Riemann-Hilbert correspondence. On the 'left-hand side' are Sakai surfaces, which provide Okamoto's initial value spaces, and on the 'right-hand side' are monodromy manifolds coming from associated Lax pairs. Symmetries of Sakai surfaces form extended affine Weyl groups and play a crucial role as Bäcklund transformation symmetries of the Painlevé equations. However, under the Riemann-Hilbert correspondence these become trivial on monodromy manifolds.

In this talk we announce results showing that there is still a shadow of the extended affine Weyl group symmetry on the monodromy manifold side, but this lies outside of the class of automorphisms of varieties. This shadow is formed by the monodromy group of the monodromy manifold itself, which we show matches exactly with the underlying finite Weyl group from the symmetries of the corresponding Sakai surface.

40. Yuri B. Suris

Title: How geometry helps to repair non-integrable Kahan discretizations

Authors: Yuri B. Suris

Institutions: Institut für Mathematik, Technische Universität Berlin, Germany.

Abstract: The bilinear discretization of quadratic vector fields, introduced by Kahan, and earlier in the context of integrable systems by Hirota (and his student Kimura), has been much studied in the recent years. It is known to preserve integrability in an amazingly large number of instances, but not always. This, a Kahan discretization of an integrable systems may happen to be non-integrable. We demonstrate that in the latter case the situation can be "repaired". Namely, one can slightly perturb the coefficients of the non-integrable Kahan discretization so that the perturbation becomes integrable. Insights into how to achieve this are delivered by an in-depth study of the underlying geometry. We will illustrate this by several two- and three-dimensional examples.

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41. Takao Suzuki

Title: A Lax formulation of a generalized q-Garnier system

Authors: Takao Suzuki

Institutions: Department of Mathematics, Kindai University, Japan.

Abstract: In a recent work [1], a birational representation of an extended affine Weyl group of type $A^{(1)}_{mn-1} \times A^{(1)}_{m-1} \times A^{(1)}_{m-1}$ was proposed with the aid of a cluster mutation. In this contribution, we formulate this representation in a framework of a system of q-difference equations with $mn \times mn$ matrices. This formulation is called a Lax form and is used to derive generalizations of the q-Painlevé equations. As a matter of fact, the obtained system implies the known systems as follows.

- $(m, n) = (2, 2)$: the q-Painlevé equation of type $D^{(1)}_5$ (Jimbo-Sakai's q-P_{VI})
- $m = 2,\ n \geq 2$: the q-Garnier system (cf. [2])
- $(m, n) = (3, 1)$: the q-Painlevé equation of type $E^{(1)}_6$

In this contribution, we also consider the case $(m, n) = (3, 2)$ as an example. Then we obtain a system of nonlinear q-difference equations of eighth order. It admits a particular solution in terms of a q-hypergeometric function and reduces to two representative isomonodromy deformation equations, the Fuji-Suzuki-Tsuda system and the Sasano system (cf. [3]), via a continuous limit. This contribution is based on the paper [4].

References:

[1] T. Suzuki and N. Okubo, Cluster algebra and q-Painlevé equations: higher order generalization and degeneration structure. *RIMS Kôkyûroku Bessatsu.* **B78**: 53–75, 2020.
[2] N. Okubo and T. Suzuki, Generalized q-Painlevé VI systems of type $(A_{2n+1} + A_1 + A_1)^{(1)}$ arising from cluster algebra. *Int. Math. Res. Not.*. **2020** (9): 6561–6607, 2020.
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42. Tomoyuki Takenawa

Title: An extension to the singular pattern approach for calculating the degree growth in N-dimensional birational dynamical systems

Authors: Tomoyuki Takenawa

Institutions: Faculty of Marine Technology, Tokyo University of Marine Science and Technology.

Abstract: For a birational map f, let the degree of its n-th iterate be $d_n = \deg(f^n)$. Then $e(f) = \lim_{n\to\infty} \frac{1}{n}\log(d_n)$ is called the algebraic entropy [1]. In this talk, we consider methods for computing the growth rate of these degrees. Traditional approaches –whether algebraic-geometric techniques based on the construction of rational varieties or detailed analyses of factorisation patterns of rational functions– both involve cumbersome procedures and often high computational cost. Halburd [2] greatly simplified the latter approach by proposing, for a single-variable high-order recurrence, a method that directly yields a recurrence for the degrees simply by observing the singularity patterns in the sense of Grammaticos–Ramani [3]. While this method is computationally light, it is limited in that it cannot be applied when the singularity pattern produces only trivial recurrences.

In this talk, we show that by treating Halburd's idea not as a single-variable recurrence but as an N-dimensional birational map, one can derive nontrivial recurrence relations for the degrees that were previously unattainable, and thus compute the algebraic entropy for a much broader class of examples.

References:

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[3] B. Grammaticos, A. Ramani, and V. Papageorgiou, Do integrable mappings have the Painlevè property? *Phys. Rev. Lett.* **67**, 1825, 1991.

43. Tetsuji Tokihiro

Title: Integrable cellular automaton on finite fields of order of power of 2

Authors: Aoi Araoka, Tetsuji Tokihiro*

Institutions: Department of Mathematical Engineering, Faculty of Engineering, Musashino University, Japan.

Type of contribution: Talk or Poster (Either is fine)

Abstract: A cellular automaton (CA) is a discrete dynamical system consisting of a set of cells with a finite number of states. It is natural to describe the state of a cell using a finite field [1]. When considering an integrable CA [2], it is natural to construct it so that its time evolution is based on the

Yang-Baxter equation. However, there have been few systematic studies to derive CAs from the Yang-Baxter equation on finite bodies [3]. Therefore, we wish to construct CAs based on an R-matrix satisfying the Yang-Baxter equation on a finite field, and to study what kind of characteristics the CA has. In this study, in particular, we consider finite fields of order of powers of two ($\mathbb{F}_{2^n}$). We give a set of solutions to Yang-Baxter equation over $\mathbb{F}_{2^n}$ and show that the corresponding CA always has period 4 for $n = 2, 3$.

References:

[1] F. Yura, Solitons with a nested structure over finite fields, J. Phys. A: Math. Theor. **47** 325201, 2014.
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[3] M. Bruschi, P. M. Santini, and O. Ragnisco, Integrable cellular automata Phys. Lett. A **169**, 151–60, 1992

44. Satoshi Tsuchimi

Title: A generalization of the μ-function and its relation to integrable systems.

Authors: Satoshi Tsuchimi

Institutions: Department of Mathematics, Kindai University, Japan.

Abstract: In this talk, we present a special solution of the type $(A_2+A_1)^{(1)}$ q-Painlevé equation associated with the μ-function. The μ-function, introduced by Zwegers in 2002, is a fundamental object in the study of the mock theta functions [3]. At the beginning of this talk, we consider the μ-function in view of q-difference equations and introduce a one-parameter deformation of the μ-function [2]. Then, we show that the generalized μ-function is closely related to the τ-function of discrete integrable systems [1].

References:

[1] S. Tsuchimi, *A solution in terms of mock modular forms for the q-Painlevé equation of the type* $(A_2 + A_1)^{(1)}$, arXiv: 2405.02902.
[2] G. Shibukawa and S. Tsuchimi, *A generalization of Zwegers' μ-function according to the q-Hermite–Weber difference equation*, SIGMA, **19** (2023) 014 pp23.
[3] S. P. Zwegers, *Mock theta functions*, Thesis, Universiteit Utrecht (2002).

45. Teruhisa Tsuda

Title: Birational Weyl group actions and q-Painlevé equations via mutation combinatorics in cluster algebras

Authors: Tetsu Masuda, Naoto Okubo, and Teruhisa Tsuda*

Institutions: Department of Mathematical Sciences, Aoyama Gakuin University

Abstract: A cluster algebra is an algebraic structure generated by operations of a quiver (a directed graph) called the mutations and their associated simple birational mappings. By using a graph-combinatorial approach, we present a systematic way to derive a tropical, i.e. subtraction-free birational, representation of Weyl groups from cluster algebras. Our result provides a broad class of Weyl group actions including previously known examples acting on certain rational varieties and hence it is relevant to q-Painlevé equations and their higher-order extensions. Key ingredients of the argument are the combinatorial aspects of the reflection associated with a cycle subgraph in the quiver. We also discuss symplectic structures of the discrete dynamical systems thus obtained.

46. Mats Vermeeren

Title: Discrete Lagrangian multiforms for ABS equations revisited

Authors: Jacob J. Richardson[1], Mats Vermeeren[2,*]

Institutions:

(1) School of Mathematics, University of Leeds, UK.
(2) Department of Mathematical Sciences, Loughborough University, UK.

Abstract: Discrete Lagrangian multiform theory is a variational perspective on lattice equations that are integrable in the sense of multidimensional consistency. The Lagrangian multiforms for the quad equations of the Adler-Bobenko-Suris (ABS) classification formed the start of this theory, but the Lagrangian multiforms that are usually considered in this context produce equations that are weaker than the ABS equations. In this work, we present alternative Lagrangian multiforms whose Euler-Lagrange equations are equivalent to the ABS equations.

In addition, we address a second issue with the treatment of the ABS Lagrangian multiforms in the existing literature: it fails to acknowledge that the complex functions in their definitions have branch cuts. The choice of branch affects both the existence of an additive three-leg form for the ABS equations and the closure property of the Lagrangian multiforms. We show that these properties can be recoverd by including integer-valued fields, related to the possible branch choices, in the action sums.

References:

[1] J. J. Richardson, and M. Vermeeren, *Discrete Lagrangian multiforms for ABS equations I: quad equations.* arXiv:2501.13012.

47. Luc Vinet

Title: Bispectrality of the sieved Jacobi polynomials

Authors: Luc Vinet[1,*], Alexei Zhedanov[2]

Institutions:

(1) CRM and IVADO, Université de Montréal
(2) School of Mathematics, Renmin University of China.

Abstract: It is shown that the CMV Laurent polynomials associated to the sieved Jacobi polynomials on the unit circle satisfy an eigenvalue equation with respect to a first order differential operator of Dunkl type. Using this result, the sieved Jacobi polynomials on the real line are found to be eigenfunctions of a Dunkl differential operator of second order. Eigenvalue equations for the sieved ultraspherical polynomials of the first and second kind are obtained as special cases. These results mean that the sieved Jacobi polynomials (either on the unit circle or on the real line) are bispectral. Based on [1].

References:

[1] arXiv:2501.12806

48. Andreas Vollmer

Title: A geometric framework for second-order maximally superintegrable Hamiltonian systems

Authors: Andreas Vollmer

Institutions: Department of Mathematics, University of Hamburg, Germany.

Abstract: This talk introduces a geometric framework for second-order superintegrable Hamiltonian systems on Riemannian manifolds in arbitrary dimension. It encodes the systems in symmetric cubic tensor fields or, equivalently, in naturally associated affine connections [3, 5, 8]. The framework also applies to conformally superintegrable systems, providing a geometric interpretation of Stäckel transformations (also known as coupling constant metamorphosis) through the lens of conformal and Weylian geometry [2, 4, 6].

The framework will be exemplified through so-called abundant systems, i.e. those with a maximal number of compatible potentials and of linearly independent integrals of motion. These include the well-known Smorodinski-Winternitz system. Novel curvature conditions are obtained for abundant systems [3, 4]. Moreover, a natural correspondence between abundant superintegrable systems and affine hypersurfaces is established [2], making Stäckel transformations special choices of transversal fields. In particular, on spaces of constant sectional curvature, abundant systems are found to have an underlying Hessian structure [1]. On flat space, we show that they correspond to solutions of the Witten-Dijkgraaf-Verlinde-Verlinde equation, allowing their identification with Manin-Frobenius manifolds [3, 7].

References:

[1] J. Armstrong, and A. Vollmer, *Abundant Superintegrable Systems and Hessian Structures*, arXiv:2410.05009

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[5] J. Kress, K. Schöbel, and A. Vollmer, *Superintegrable systems on conformal surfaces*, arXiv:2403.09191
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[8] A. Vollmer, *Torsion-free connections of second-order maximally superintegrable systems*, Bull. London Math. Soc. 57(2), 2025

49. Ralph Willox

Title: Dynamical degrees for non-confining and higher order mappings from deautonomisation by singularity confinement

Author: B. Grammaticos[1], A. Ramani[1], and Ralph Willox[2]

Institution:

(1) Laboratoire de Physique des 2 Infinis Irène Joliot-Curie;
(2) Graduate School of Mathematical Sciences, the University of Tokyo.

Abstract: Deautonomisation by singularity confinement has been shown to be a powerful tool for obtaining the dynamical degrees for mappings of order two that only possess confining singularities [1]. In this talk we will explain how this method can also be used for higher order mappings, be they integrable or not, as well as for mappings of order two that have singularities which do not have the confinement property.

References:

[1] A. Stokes, T. Mase, R. Willox and B. Grammaticos, Deautonomisation by singularity confinement and degree growth, *J. Geom. Anal.* **35**, 65, 2025.

50. Sikarin Yoo-Kong

Title: The q-deformed Calogero's Goldfish model

Authors: Umpon Jairuk[1], Thanadon Kongkoom[2], and Sikarin Yoo-Kong[2,*]

Institutions:

(1) Division of Physics, Faculty of Science and Technology, Rajamangala University of Technology Thailand;

(2) The Institute for Fundamental Study, Naresuan University, Thailand.

Abstract: Searching for integrable models is a central theme in theoretical and mathematical physics, as such systems offer valuable insights into the underlying structure and symmetries of complex physical phenomena. In this work, we contribute to this pursuit by proposing a new class of one-dimensional many-body integrable systems, which we refer to as the q-deformed Calogeros Goldfish system. Our construction employs q-deformation of logarithmic and exponential functions inspired by Tsallis formalism in non-extensive statistical mechanics. Notably, the model satisfies the double-zero condition on its solutions, underscoring its integrable nature and offering a novel perspective on deformation techniques within exactly solvable systems.

51. Da-jun Zhang

Title: Eigenfunction and discrete integrable systems

Authors: Leilei Shi, Cheng Zhang, and Da-jun Zhang*

Institutions: Department of Mathematics, Shanghai University, People's Republic of China.

Abstract: We consider the discrete Schrödinger spectral problem related to the discrete potential KdV equation (H1 in the ABS list). Eigenfunctions of this spectral problem and their certain combinations satisfy some discrete equations, which we call eigenfunction equations. We explain the connection of these equations and some ABS equations. We also consider two D-bar problems associated with given discrete plane wave factors. We show how discrete integrable equations arise from the D-bar approach. Finally, we connect the functions defined from the eigenfunctions, D-bar problems and direct linearisation approach.

52. Yao-Zhong Zhang

Title: Yang-Baxter integrability, R-matrices and exactly solvable lattice models

Authors: Jason Werry, Yao-Zhong Zhang*

Institutions: School of Mathematics and Physics, University of Queensland, Brisbane Australia.

Abstract: Quantum deformations of (affine) Lie (super) algebras are symmetry algebras of quantum integrable lattice models (such as spin chains) in the Yang-Baxter sense, and have applications in many branches of mathematics and physics. This talk will describe how to apply the quantized algebraic structures to construct solutions of the Yang-Baxter equation and thus the corresponding integrable lattice models. We give general formulas for multivariate R-matrices (with additive and non-additive spectral parameters, respectively) associated with a type-I quantum superalgebra. The spectral parameters can either originate from the loop parameter the quantum affine

superalggebra or the one-parameter families of inequivalent finite-dimensional irreducible representations of the quantum superalgebra upon which the R-matrix acts. Applying to the quantum affine superalgebra $U_q(\widehat{gl(2|1)})$, we give the explicit expressions for the $U_q(gl(2|1))$-invariant R-matrices. These R-matrices lead to new two-parameter integrable models of strongly correlated electrons, giving integrable one-parameter extension of Hubbard and Barieve models. This talk is based on [1].

References:

[1] Y-Z. Zhang, and J. L. Werry, New R-matrices with non-additive spectral parameters and integrable models of strongly correlated fermions, *J. High Energy Phys.* **2020**, 1–13, 2020.

Posters

In the description of the talks the $*$ marks the presenting author.

1. Shannon Abbott

Title: On a fixed point method for a discrete Painleve II equation

Authors: Shannon Abbott

Institutions: School of Engineering, Mathematics and Physics University of Kent, UK.

Abstract: Recently Van Assche [1] showed the existence and uniqueness of a special solution of a discrete Painleve II equation, which arises in connection with certain orthogonal polynomials on the unit circle, and are expressed explicitly in terms of a ratio of modified Bessel functions. Here we consider a fixed point iteration on semi-infinite sequences of real numbers in the interval $[-1, 1]$, which converges to this unique solution of discrete Painleve II, and provides an effective numerical scheme for computing these classical solutions.

References:

[1] W. Van Assche, Unique special solution for discrete Painlevé II, *J. Diff. Equ. Appl.* **30**, 465-474, 2024.

2. Kanae Akaiwa

Title: Inverse Eigenvalue Problem for Periodic Tridiagonal Matrices via Discrete Toda Equation

Authors: Kanae Akaiwa

Institutions: Faculty of Information Science and Engineering, Kyoto Sangyo University, Japan.

Abstract: The discrete Toda equation, a typical discrete integrable system, corresponds to the eigenvalue problem for a tridiagonal diagonal matrix [3, 4]. One of the problems in numerical linear algebra, the inverse eigenvalue problem (IEP), involves constructing a matrix with prescribed eigenvalues. The difficulty of the problem depends on the form and properties of the matrix to be constructed [1]. Ferguson [2] proposed an algorithm for the IEP of periodic Jacobi matrices using the Lanczos method.

In this talk, based on Ferguson's ideas, we propose how to solve the IEP of constructing a periodic tridiagonal matrix, which is a generalization of the periodic Jacobi matrix, using the discrete Toda equation.

References:

[1] M. T. Chu, and G. H. Golub, *Inverse Eigenvalue Problems: Theory, Algorithms, and Applications.* Oxford University Press, New York, 2005.

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3. Debora Choinska

Title: On solutions of integrable discretizations of Ernst-type equations

Authors: Debora Choinska*, Maciej Nieszporski

Institutions: Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw.

Abstract: Ernst-type equations are elegant reformulations of Einsteins vacuum equations of general relativity when the existence of two commuting Killing vector fields is assumed. Axisymmetric, stationary spacetimes such as rotating black holes and planar gravitational waves are examples of solutions of the Ernst-type equations.

An essential mathematical property of the Ernst-type equations are that they are integrable nonlinear differential equations, in particular, there exists the nonlinear superposition principle for their solutions.

A key focus is the discretizations of the Ernst-type equation (i.e. difference equations that in a continuum limit become Ernst-type equations) that exhibit all the features of integrability including the nonlinear superposition principle. In the literature, one can find two discretisations of this kind.

I will present exact solutions to the discrete equations.

4. Susanna Dehò

Title: Rotational invariance of integration by parts formula and Lie symmetries of SDEs

Authors: Susanna Dehò[1,*], F.C. De Vecchi[2], P. Morando[3], and S. Ugolini[1].

Institutions:

(1) Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano, Italy;
(2) Dipartimento di Matematica, Università degli Studi di Pavia, Italy;
(3) Dipartimento di Scienze Agrarie e Ambientali - Produzione, Territorio, Agroenergia, Università degli Studi di Milano, Italy.

Abstract: The study of symmetry properties of ordinary differential equations and partial differential equations is a classical and well-established topic in the literature, offering both a powerful tool for the explicit computation of solutions and a deeper understanding of their qualitative behavior. In contrast, a theory of symmetries for stochastic differential equations (SDEs), analogous to the deterministic case, has been developed only in recent years [1, 2], also highlighting significant connections with the symmetries of the associated Fokker-Planck or Kolmogorov equations, see [3–5]. The application of Lie symmetry theory to SDEs enables the derivation of integration by parts formulas inspired by Bismuts variational approach to Malliavin calculus, with notable applications to the analysis of the law and regularity of the processes, as well as to the development of a stochastic calculus of variations, see [6, 7].

Various notions of invariance properties and symmetries for SDEs, including strong, weak, and gauge symmetries, will be described here, with particular emphasis on the rotational invariance of the driving Brownian motion and the associated infinitesimal generator. The stochastic rotational invariance of the integration by parts formula proposed in [8] will also be demonstrated and discussed through applications to selected Brownian motion-driven stochastic models [9].

References:

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[2] S. Albeverio, F. C. De Vecchi, P. Morando, and S. Ugolini, *Random transformations and invariance of semimartingales on Lie groups*, Random operators and Stochastic Equations, (2021).
[3] F. C. De Vecchi, P. Morando, and S. Ugolini, *Symmetries of stochastic differential equations using Girsanov transformations*, *Journal of Physics A: Mathematical and Theoretical*, **53** (2020), no. 13, 135204.
[4] R. Kozlov, *Symmetries of Kolmogorov backward equation*, *Journal of Nonlinear Mathematical Physics*, **28** (2021), 182–193.
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[9] S. Dehò, F. C. De Vecchi, P. Morando, and S. Ugolini, *Random rotational invariance within a Bismut-type approach to integration by parts formulas*, Preprint 2025.

5. Marta Dell'Atti

Title: Exploring quartic Hamiltonian systems of Painlevé and quasi-Painlevé type

Authors: Marta Dell'Atti[1,*], Thomas Kecker[2]

(1) Institute of Mathematics, University of Warsaw, Poland;
(2) School of Mathematics and Physics, University of Portsmouth, UK.

Abstract: We investigate non-autonomous Hamiltonian systems that are quartic in the dependent variables, focusing on the local behavior of solutions around movable singularities. These singularities include simple poles (Painlevé type) and algebraic poles (quasi-Painlevé type), with the systems encompassing both cases where singularities are of the same type and cases where they are of mixed types.

Using a geometric approach, we assign a surface type to each system by constructing the space of initial conditions (for Painlevé systems) or its analogue as a defining manifold (for quasi-Painlevé systems). The classification is based on the initial base points in the extended phase space ($\mathbb{CP}^2$) and their multiplicities, which arise from the coalescence of four simple base points in the generic case. Through successive degeneration and further coalescence of base points, we derive all possible sub-cases of quartic Hamiltonian systems with the quasi-Painlevé property. These sub-cases are finally characterized by Newton polygons associated with the polynomial Hamiltonians, the types of singularities, and the corresponding surface types.

This multi-dimensional description enables a comprehensive classification of the systems. As particular examples, we recover systems equivalent to known Painlevé equations. The talk is based on the works [1, 2].

References:

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[2] M. Dell'Atti, and T. Kecker, Spaces of initial conditions for quartic Hamiltonian systems of Painlevé and quasi-Painlevé type. Preprint arXiv:2412.17135, 2025.

6. Pavel Drozdov

Title: Explicit isomorphisms for the symmetry algebras of isotropic oscillators

Authors: Pavel Drozdov[1,*], Giorgio Gubbiotti[2], Danilo Latini[2]

Institutions:

(1) Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università degli Studi di Udine, Udine, Italy and INFN Sezione di Trieste, Trieste, Italy,

(2) Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano and INFN Sezione di Milano, Milan, Italy.

Abstract: In classical and quantum mechanics, a Demkov–Fradkin tensor is a quantity that encompasses enough integrals of motion to make the isotropic harmonic oscillator (IHO) [6, 7] in N degrees of freedom maximally superintegrable (MS) [3, 4]. The components of Demkov–Fradkin tensor and angular momentum form a Lie algebra with respect to the canonical Poisson bracket $\{q_i, p_j\} = \delta_{i,j}$ for all values of the energy. We consider it as a parametric family of real Lie algebras $\mathfrak{A}_N(\alpha)$, where $\alpha \in \mathbb{R}$ represents the frequency of the IHO when it is positive.

Throughout the years the Demkov–Fradkin tensor has been extended also to many nonlinear MS systems [2, 8]. Motivated by the appearance of the Lie algebra in problems related to various models, in the work [5], we performed a comprehensive study of the algebra $\mathfrak{A}_N(\alpha)$ for arbitrary N and proved **explicit** formulæ, establishing the following isomorphisms:

$$\mathfrak{A}_N(\alpha) \cong \begin{cases} \mathfrak{u}_N, & \alpha > 0, \\ \mathfrak{so}_N(\mathbb{R}) \ni \mathbb{R}^{\mathcal{N}}, & \alpha = 0, \\ \mathfrak{gl}_N(\mathbb{R}), & \alpha < 0, \end{cases}$$

where $\mathbb{R}$ is the one-dimensional abelian Lie algebra [1] and $\mathcal{N} \coloneqq N(N+1)/2$.

Finally, we will also discuss some applications to IHO discretization and other nonlinear systems.

References:

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7. Shuhei Kamioka

Title: Random sampling of plane partitions based on integrable systems

Authors: Shuhei Kamioka

Institutions: Osaka Seikei University, Osaka, Japan

Abstract: Plane partitions are combinatorial objects which can be exactly enumerated, and nice enumeration formulas and generating functions, such as MacMahon's formula, are known (a nice survey is given by Krattenthaler [2]).

Plane partitions exhibit intriguing properties from probabilistic and statistical perspectives. For example, when randomly sampling a huge plane partition satisfying a certain boundary condition in accordance with the uniform distribution, we observe a specific limit shape known as the arctic ellipse [1]. To investigate such probabilistic properties of plane partitions, algorithms for random sampling are essential. In particular, for sampling huge plane partitions to observe limit shapes, the speed of algorithms is of critical importance. Moreover, the ability to handle a variety of probability distributions beyond the uniform case is expected to be beneficial for exploring unknown probabilistic properties.

In this presentation, we introduce an algorithm for randomly sampling plane partitions based on the discrete Toda equation. This algorithm is computationally so fast as to generate a plane partition in time proportional to its scale. Furthermore, it accommodates non-uniform probability distributions, where the distributions are determined by specific solutions to the equation.

References:

[1] H. Cohn, M. Larsen, and J. Propp, The shape of a typical boxed plane partition. *New York J. Math.*. **4**: 137–165, 1998.
[2] C. Krattenthaler, Plane partitions in the work of Richard Stanley and his school, in: *The Mathematical Legacy of Richard P. Stanley*, American Mathematical Society, 2016, pp. 231–261.

8. Danilo Latini

Title: TBA

Authors: Danilo Latini

Institutions: Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano and INFN Sezione di Milano, Milan, Italy.

Abstract: TBA

9. Juho Halonen

Title: A reduction of the ultradiscrete modified KdV equation and its finite order solutions

Authors: Juho Halonen

Institution: Department of Physics and Mathematics, University of Eastern Finland, Finland.

Abstract: In this talk we will present a reduction of the ultradiscrete modified KortewegDe Vries equation and give a detailed description of its solution. We will also show that with certain parameter values of the reduction, it contains only finite order solutions in the sense of tropical Nevanlinna theory, while for other parameter values the reduction contains infinite order solutions.

10. Harry Hiatt

Title: Billiard Knots

Authors: Harry Hiatt*, and Milena Radnovič

Institutions: School of Mathematics and Statistics, The University of Sydney.

Abstract: This poster is about the knotting of billiard trajectories inside 3-dimensional billiard domains. The system under consideration consists of a single point like billiard ball that reflects elastically off the sides of the billiard table, and we are interested in studying its periodic, non-intersecting billiard trajectories. We present results concerning the obtained knots within certain billiard domains.

11. Kohei Higashi

Title: Discretization of Nonlinear Integrable Systems with Singular Integral Terms

Authors: Kohei Higashi

Institutions: Faculty of Engineering, Musashino University, Tokyo, Japan.

Abstract: Nonlocal effects play a pivotal role in mathematical models across fluid mechanics, traffic flow, and neural field theory.

Within these models, certain equations are known to form integrable systems and have been shown to appear as boundary values under an analytic reduction imposed on equations defined in complex domains [1].

In this work, we develop a discretization approach for these equations through the framework of discrete integrable systems and the discrete RiemannHilbert problem. We also derive exact solutions and examine their properties.

References:

[1] Kohei Higashi, *Non-local time evolution equation with singular integral and its application to traffic flow model*, Phys. Scr. **99**, 085212, 2024.

12. Techheang Meng

Title: Analytic continuation of solutions of Difference equations

Authors: Techheang Meng

Institutions: University College London (UCL), London, United Kingdom.

Type of contribution: Poster.

Abstract: It is well known that integrability and singularity structure are closely connected. We will look at the special solutions of the difference equation $y(z+1) = R(y(z))$, where R is a rational function, that, in general, is not integrable. It has been shown that when R has a repelling fixed point, there exists a solution parametrised in terms of a Poincaré function, which can be analytically continued into a meromorphic function. A similar situation is known around an attracting fixed point of R, however, such a solution admits branch points. In this case, we will see that $y(z)$ can be analytically continued to an infinite-sheeted Riemann surface. We present a detailed example with $R(w) = \lambda w + w^2$, where $0 < \lambda < 1$, to illustrate how complicated the singularity structure of the special solution can be, using simple arguments from complex analysis. When R is a higher degree rational function, we give additional examples to show how diverse the distribution of the singularities of the Riemann surface of the special solutions are. Furthermore, we also show how different the singularities can be when constructing special solutions near neutral or super-attracting fixed points of R.

References:

[1] I.N. Baker, and L.S.O. Liverpool, The entire solutions of a polynomial difference equation, Aequationes mathematicae, vol. 27, pp. 97–113, 1984.

[2] R. Halburd, R. Korhonen, Y. Liu, and T. Meng, On the extension of analytic solutions of first-order "difference" equations, `arXiv, nlin.SI, https://arxiv.org/abs/2502.03955`.

13. H. W. A. Riaz

Title: Darboux transformation for a semi-discrete matrix coupled dispersionless system

Authors: J. Lin, H. W. A. Riaz*

Institutions: Department of Physics, Zhejiang Normal University, 321004, PR China

Abstract: We present a semi-discrete matrix coupled dispersionless system. We propose a Lax pair and employ the Darboux transformation to construct explicit solutions. These solutions reveal diverse phenomena, including periodic patterns, breathers, and rogue waves. Notably, the system exhibits intricate interactions between bright and dark solitons and between bright solitons and rogue waves. Furthermore, we provide a numerical experiment for the reduced semi-discrete coupled dispersionless equations to demonstrate the effectiveness of the developed method.

14. Takiko Sasaki

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Title: A blow-up theorem for discrete semilinear wave equation

Authors: Kohei Higashi, Keisuke Matsuya, Takiko Sasaki*, Tetsuji Tokihiro, Ryosuke Tsubota

Institutions: Department of Mathematical Engineering, Faculty of Engineering, Musashino University

Abstract: In this talk, we evaluate lifespan of the solution of the equation obtained by discretising a semilinear wave equation with a power-type nonlinear term. The discrete equation was first proposed by Matsuya [1] and has the following form.

$$u_n^{t+1} + u_n^{t-1} = \frac{4v_n^t}{2 - \delta^2 v_n^t |v_n^t|^{p-2}} \quad (n \in \mathbb{Z}^d,\, t \in \mathbb{Z}_{\geq 0})$$

$$v_n^t := \frac{1}{2d} \sum_{i=1}^{d} \left(u_{n+e_i}^t + u_{n-e_i}^t \right)$$

In continuous limit, this discrete equation turns to the semiliniar wave equation:

$$u_{tt} = \Delta u + |u|^p$$

This semilinear wave equation is known to explode if the exponent p appearing in the nonlinear term is smaller than a certain value when the initial conditions are sufficiently small, and the discrete equation has been proved to have similar behaviour to the original wave equation. We show that the discrete equation also has similar lifespan to that of the semilinear wave equation.

References:

[1] Keisuke Matsuya, A blow-up theorem for a discrete semilinear wave equation, *J. Diff. Eq. Appl.*, **19** (3), 457–465, 2013.

15. Edoardo Peroni

Title: Quadratic Darboux transformations for the non-commutative DNLS equations. Reductions and associated differential-difference equations

Authors: Edoardo Peroni [1,*], Jing Ping Wang[1,2]

Institutions:

(1) School of Engineering, Mathematics and Physics University of Kent, UK;
(2) School of Mathematics and Statistics, Ningbo University, Peoples Republic of China.

Abstract: In the commutative framework, the term *derivative non-linear Schrödinger* (DNLS) equations refer to a family of integrable equations, related by a gauge transformation, including Kaup-Newell, Chen-Lee-Liu and Gerdjikov-Ivanov equations. The non-commutative analogues of the DNLS equations were identified by Olver and Sokolov [1], consisting in seven integrable PDEs with Lax representations provided by Tsuchida and Wadati [2]. In this talk, we construct the quadratic Darboux matrices associated with the non-commutative DNLS. We establish connections with integrable differential-difference equations (DΔE) involving non-commuting constants, leading to generalisations of known integrable lattices. The current work extends and completes previous studies [3, 4].

References:

[1] P. J. Olver, and V. V. Sokolov, *Non-abelian integrable systems of the derivative nonlinear Schrödinger type.* Inverse Problems, 14(6):L5, 1998.
[2] T. Tsuchida, and M. Wadati, *Complete integrability of derivative nonlinear Schrödinger-type equations.* Inverse Problems, 15(5):1363, 1999.
[3] S. Konstantinou-Rizos, and A. V. Mikhailov, and P. Xenitidis, *Reduction groups and related integrable difference systems of nonlinear Schrödinger type* Journal of Mathematical Physics, 56(8):082701, 2015.
[4] S. Konstantinou-Rizos, and A. A. Nikitina, *YangBaxter maps of KdV, NLS and DNLS type on division rings.* Physica D: Nonlinear Phenomena, 465:134213, 2024.

16. Dmitrii Rachenkov

Title: Polynomial duality

Authors: Dmitrii Rachenkov

Institutions: Scuola Internazionale Studi Superiori Avanzati, Trieste, Italy

Abstract: The *Shapiro-Tater conjecture* [1], that was recently proved in [2], is about coinsidence of zeros of two different families of polynomials:
Vorob'ev-Yablonskii polynopmials $Y_n(t)$ and *discriminant polynomials* $D_n(t)$, under the limit $n \to \infty$. Here $Y_n(t)$ are polynomials that generates rational solutions of the Painlevé equation $\mathrm{P_{II}}$ whilst $D_n(t)$ describe degeneration of quasi-rational spectrum of anharmonic oscillator with quartic potential.

In my talk I am going to present the analog of the conjecture for the case of the Painlevé equation P_{IV}.

References:

[1] B. Shapiro and M. Tater, On spectral asymptotic of quasi-exactly solvable quartic. *Anal. Math. Phys.* 12, **2** (2022).
[2] M. Bertola, E. Chavez-Heredia and T. Grava, Exactly Solvable Anharmonic Oscillator, Degenerate Orthogonal Polynomials and Painlevé II. *Commun. Math. Phys.* 405, **52** (2024).

17. Emanuele Sgroi

Title: Lie algebras with compatible scalar products for non-homogeneous Hamiltonian operators

Authors: Giorgio Gubbiotti[1,2], Francesco Oliveri[3], Emanuele Sgroi[4,*], Pierandrea Vergallo[2,4]

Institutions:

(1) Università degli Studi di Milano, Dipartimento di Matematica "Federigo Enriques", Via Cesare Saldini 50, 20133, Milano, Italy,
(2) INFN Sezione di Milano, Via Giovanni Celoria 16, 20133, Milano, Italy,
(3) Università degli Studi di Messina, Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra, V.le F. Stagno D'Alcontres 31, I-98166 Messina, Italy,
(4) Università degli Studi di Messina, Dipartimento di Ingegneria, Contrada Di Dio, 98166 Sant'Agata, Messina, Italy.

Abstract: In this talk, we investigate Hamiltonian structures composed by a non-degenerate first-order homogeneous operator and a Poisson tensor. in flat coordinates, these operators are uniquely determined by a Lie algebra, a non-degenerate quadratic Casimir, and a 2-cocycle. We explore their algebraic and geometric structure and provide a complete description up to dimension six. Using a purely algebraic approach, we study abelian and semisimple Lie algebras, direct sum structures, and the particular case of two-step nilpotent Lie algebras, describing the classes and some related examples [1].

References:

[1] Giorgio Gubbiotti, Francesco Oliveri, Emanuele Sgroi and Pierandrea Vergallo, Lie algebras with compatible scalar products for non-homogeneous Hamiltonian operators, 2025, arXiv preprint `arXiv:2502.05137`.

18. Chen Shu

Title: A New Transform for the Solutions of Moutard-Transformable Integrable Systems

Authors: Claire Gilson, Chen Shu[*]

Institutions: School of Mathematics & Statistitics, University of Glasgow.

Abstract: We present a new transform for generating solutions of Moutard-transformable integrable systems, which is possible to extend to the noncommutative case. This transform is based on a novel structure called the quasi-Pfaffian, which is analogous to the quasi-determinant [2] and its Sylvester identity [3]. As an example, we demonstrate the application of this transform to the com- mutative case of the Novikov-Veselov system, building upon previous results from [1].

References:

[1] C. Athorne and J.J.C. Nimmo. On the Moutard transformation for integrable partial differential equations. In: *Inverse problems* **7**.6 (1991), p. 809.

[2] C.R. Gilson and J.J.C. Nimmo. On a direct approach to quasideterminant solutions of a noncommutative KP equation. In: *Journal of Physics A: Mathematical and Theoretical* **40**.14 (2007), p. 3839.

[3] J.J. Sylvester. XXXVII. On the relation between the minor determinants of linearly equivalent quadratic functions. In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 1.4 (1851), pp. 295–305.

19. Pierandrea Vergallo

Title: Bi-Hamiltonian aspects of non-homogeneous operators

Authors: Marta Dell'Atti[1], Alessandra Rizzo[2], Pierandrea Vergallo[3,*]

Institutions:

(1) Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Poland ;
(2) Department of Mathematics, University of Palermo, Italy;
(3) Department of Engineering, University of Messina, Italy.

Type of contribution: Poster.

Abstract: In the 80s, Dubrovin and Novikov introduced for the first time Poisson brackets with the homogeneity property in the order of derivation, revealing many applications in mathematical physics and an effective interpretation in geometry. As a first example, they showed that first-order homogeneous Hamiltonian operators play a fundamental role in the theory of intregrable systems, also for their clear geometric interpretation in Riemannian geometry. Furthermore in [1], the authors extended this structure by adding a non-homogeneous term of degree zero, i.e. an ultralocal structure, which is naturally associated to non-homogeneous quasilinear systems of first-order PDEs.

In this talk, we present a complete classification of non-homogeneous $1+0$ operators in $n=2$ and $n=3$ number of components, covering the case of the Hamiltonian structure for the *inverted* KdV equation. Starting by this leading example, we then discuss a bi-Hamiltonian formalism in which both

the operators are non-homogeneous, showing a classification of compatible pairs in $n = 2$ components. The resulting Poisson brackets reveal an interesting geometric interpretation in terms of Nijenhuis geometry and bi-pencils of compatible metrics jointly with ultralocal structures. We finally discuss non-homogeneous quasilinear systems admitting this formalism, showing necessary conditions of compatibility between the operators and the investigated systems.

This poster is based on [2–4].

References:

[1] Dubrovin B A, Novikov S P: *On Poisson brackets of the hydrodynamic type*. Akad. Nauk SSSR Dokl. 279:2 pp. 294–297 (1984).

[2] Dell'Atti M, Vergallo P: *Classification of degenerate non-homogeneous Hamiltonian operators*. J. Math. Phys. 64:3 (2022). DOI: https://doi.org/10.1063/5.0135134

[3] Vergallo P.: *Non-homogeneous Hamiltonian structures for quasilinear systems*. Boll. Unione Mat. Ital., pp. 114 (2024). DOI: https://doi.org/10.1007/s40574-023-00369-5

[4] Dell'Atti M, Rizzo A, Vergallo P: *Geometric aspects of non-homogeneous 1+0 operators*, arxiv:2503.21917 (2025).

20. Jianghao Xu

Title: Galilean symmetry of the KdV hierarchy

Authors: Jianghao Xu*, Di Yang

Institution: School of Mathematical Sciences, University of Science and Technology of China,People's Republic of China;

Abstract: By solving the infinitesimal Galilean symmetry for the KdV hierarchy, we obtain an explicit expression for the corresponding one-parameter Lie group, which we call the *Galilean symmetry* of the KdV hierarchy [1]. As an application, we establish an explicit relationship between the *non-abelian Born–Infeld partition function* and the *generalized Brézin–Gross–Witten partition function*.

References:

[1] Xu, J., Yang, D.: Galilean symmetry of the KdV hierarchy. J. Lond. Math. Soc.(2) **111** (2025), no. 2, Paper No. e70075.

21. Junze Zhang

Title: The Geometry of Non-Commutative Integrable Systems and Their Algebraic Construction

Authors: Junze Zhang

Institutions: School of Mathematics and Physics, University of Queensland, Brisbane, Australia.

Abstract: Non-Commutative integrable systems, often referred to as superintegrable systems, possess considerable importance in different fields such as Poisson geometry, symplectic geometry, Hamiltonian mechanics, and quantum physics. This significance arises from their rich algebraic and geometric characteristics [1, 2]. Such systems are defined by having a greater number of integrals of motion compared to the number of degrees of freedom available within the system. This gap enables significant insights through algebraic models and geometric views. An especially notable aspect of these systems is their intricate linkage to the concept of symmetry. This relationship is elegantly elucidated by Noether's theorem, which explicates the correspondence between conserved quantities in dynamical systems and the intrinsic symmetries that underpin them. Traditionally, constructing superintegrable system Hamiltonians has extensively involved differential operator realizations, homogeneous space representations, and Marsden-Weinstein reduction techniques. Central to these geometric approaches is the concept of the momentum map, a pivotal tool that encodes symmetries as geometric objects and facilitates the reduction of complexity in mechanical systems through reduction chains, systematically simplifying the underlying phase spaces.

In this talk, I will focus on both algebraic and geometric approaches on construction non-commutative integrable system from different manifolds. Geometrically, we have introduced novel aspects such as magnetic geodesic flows and dynamics influenced by Lorentz forces, expanding the classical framework [3]. Parallel to these traditional geometric techniques, an algebraic approach for the formulation of superintegrable systems is emphasized by the role of subalgebras within Lie algebras and their corresponding universal enveloping algebras. This algebraic perspective furnishes a more efficient methodology for the explicit determination of superintegrable systems alongside their correlated Poisson algebras. Furthermore, I intend to deliver a classification of the generators of the Cartan centralizer corresponding to complex semisimple Lie algebras that are non-exceptional, positioned within their symmetric algebras [4–6].

References:

[1] N. Reshetikhin, Degenerately integrable systems, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 2015

[2] I. Marquette, J. Zhang and Y-Z. Zhang, Algebraic structures and Hamiltonians from the equivalence classes of 2D conformal algebras, Annals of Physics 169998 (2025)

[3] H. Lü, I. Marquette, J. Zhang and Y-Z. Zhang, Superintegrable systems in sympletic reductive spaces with magnetic geodesic flows. In preparation.

[4] R. Campoamor-Stursberg, D. Latini, I Marquette, J. Zhang, Y-Z. Zhang, Superintegrable systems associated to commutants of Cartan subalgebras in enveloping algebras, arxiv 2406.01958

[5] R. Campoamor-Stursberg, D. Latini, I Marquette, J. Zhang, Y-Z. Zhang, On the construction of polynomial Poisson algebras: a novel grading approach, arXiv:2503.03490
[6] R. Campoamor-Stursberg, D. Latini, I Marquette, J. Zhang, Y-Z. Zhang, Polynomial algebra from Lie algebra reduction chain $su(4) \supset su(2) \times su(2)$: The supermultiplet model, arXiv:2503.04108

22. Zejun Zhou

Title: On an infinite commuting ODE system associated to a simple Lie algebra

Authors: Di Yang[1], Cheng Zhang[2], Zejun Zhou[1,*]

Institutions:

(1) School of Mathematical Sciences, University of Science and Technology of China,People's Republic of China;
(2) Department of Mathematics, Shanghai University, People's Republic of China.

Abstract: Inspired by a recent work of Dubrovin [1], for each simple Lie algebra $\mathfrak{g}$, we introduce an infinite family of pairwise commuting ODEs and define their τ -functions. We show that these τ -functions can be identified with the τ -functions for the Drinfeld Sokolov hierarchy of $\mathfrak{g}$-type

References:

[1] B. Dubrovin (2019). Approximating tau-functions by theta-functions. *Communications in Number Theory and Physics*, 13(1):203–223.

23. Federico Zullo

Title: Unified structures for solutions of Painlevé equation II

Authors: Federico Zullo

Institutions: DICATAM, Università degli Studi di Brescia, Italy & Istituto Nazionale di Fisica Nucleare Sezione di Milano Bicocca, Italy.

Type of contribution: Talk.

Abstract: We present certain general structures related to the solutions of Painlevé equation II and to the solutions of the differential equation satisfied by the corresponding Hamiltonian equations, together with the tau functions. We will present a set of bilinear differential equation satisfied by the tau functions and explicit rational expressions linking the solutions of Painlevé equation II, Painlevé equation XXXIV and the Hamiltonians with the tau functions. Also, recursions for the tau functions and the Hamiltonians will be given. Also, we consider a degenerate case solvable in terms of Weierstrass elliptic functions. Properties of the Yablonskii-Vorob'ev polynomials, of the Airy-type solutions and of the more general transcendental case are all described within the same framework.

Detailed program

Sunday 15 June 2025

Arrival day.

18:00–19:00: Registration.

Monday 16 June 2025

8:30–9:30: Registration.
9:30–10:00: Inaugural address.
Morning session. Chairman: Giorgio Gubbiotti
10:00–10:30: Jarmo Hietarinta
Solutions to the constant Yang-Baxter equation: additive charge conservation in three dimensions
10:30–11:00: *Coffee break*
11:00–11:30: Frank Nijhoff
On the elliptic lattice Korteweg-de Vries equation: a curious discrete integrable system
11:30–12:00: Ian Marquette
Superintegrable deformations of Lotka-Volterra systems : A commutant approach
12:00–12:30: Pavlos Kassotakis
On quadrirational pentagon maps
12:30–14:00: *Lunch break*
First afternoon session. Chairman: Maciej Nieszporski
14:00–14:30: Roman Kozlov
Symmetries and first integrals of variational ODEs with delay
14:30–15:00: Linyu Peng
Symmetries of differential-difference equations and Noether's theorems
15:00–15:30: Nouf Alrubea
Symmetry analysis of the geodesic equations of the canonical connection on an n-dimensional Lie algebra with a co-dimension 2 abelian nilradical
15:30–16:00: *Coffee break*
Second afternoon session. Chairman: Jarmo Hietarinta
16:00–16:30: Tetsuji Tokihiro
Integrable cellular automaton on finite fields of order of power of 2

16:30–17:00: Da-jun Zhang
Eigenfunction and discrete integrable systems
17:00–17:30: Luc Vinet
Bispectrality of the sieved Jacobi polynomials

Tuesday 17 June 2025

First morning session. Chairwoman: Sandra Carillo
9:00–9:30: Xing-Biao Hu *(online)*
A Discrete Integrable System Associated with Bivariate Orthogonal Polynomials on Genus-Two Hyper-Elliptic Curves
9:30–10:00: Xiangke Chang *(online)*
On peakon flows of Novikov-type equations
10:00–10:30: Yousuke Ohyama *(online)*
Connection problem of q-linear equations and the q-Painlevé fifth equation
10:30–11:00: *Coffee break*
Second morning session. Chairman: Adam Doliwa
11:00–11:30: Yuri B. Suris
How geometry helps to repair non-integrable Kahan discretizations
11:30–12:00: Bao-Feng Feng
Integrable discrete massive Thirring model
12:00–12:30: Maciej Nieszporski
Integrable discretization of chiral models and discrete analytic functions theories
12:30–14:00: Lunch break
First afternoon session. Chairman: Yuri B. Suris
14:00–14:30: Wolfgang Schief
Integrability vs consistency on"hexagonal lattices"
14:30–15:00: Claire R. Gilson
Frieze Patterns, Wronskians and Discrete integrable systems
15:00–15:30: Sergey Smirnov
Darboux formulas for linear hyperbolic equations in discrete case
15:30–16:00: *Coffee break*
Second afternoon session. Chairman: Pierandrea Vergallo
16:00–16:30: Anup Anand Singh *(online)*
Lagrangian multiforms for finite-dimensional integrable hierarchies
16:30–17:00: Irfan Mahmood *(online)*
Noncommutative Painlevé second equation and its associated Toda chains
17:00–17:30: Galina Filipuk *(online)*
The Painlevé equivalence problem for a constrained 3D system
17:30–18:00: *Poster presentation session*
18:00–19:00: free time.
Poster session: Entrance of Palazzo dei Congressi
19:00–21:00: Free discussion with poster presenters.

Wednesday 18 June 2025

First morning session. Chairman: Anton Dzhamay

9:00–9:30: Wookyung Kim
Discrete integrable systems associated with the deformation of cluster maps of type D

9:30–10:00: Davide Dal Martello
Painlevé VI, symmetries, and clusters

10:00–10:30: Teruhisa Tsuda
Birational Weyl group actions and q-Painlevé equations via mutation combinatorics in cluster algebras

10:30–11:00: *Coffee break*

Second morning session. Chairman: Tomoyuki Takenawa

11:00–11:30: Hidetaka Sakai
Symplectic structure of discrete Painlevé equations

11:30–12:00: Kanam Park
A symmetry of a 3×3 *Lax form for the q-Painlevé equation of type* $E_6^{(1)}$

12:00–12:30: Satoshi Tsuchimi *A generalization of the* μ*-function and its relation to integrable systems*

12:30–14:00: *Lunch break*

First afternoon session. Chairwoman: Claire R. Gilson

14:00–14:30: Anton Dzhamay
Discrete Painlevé Equations on the $D_5^{(1)}$ *Sakai surface*

14:30–15:00: Yang Shi
New symmetries of discrete Painlevé equations from Normalizer theory of Coxeter groups

15:00–15:30: Takao Suzuki
A Lax formulation of a generalized q-Garnier system

15:30–16:00: *Coffee break*

Second afternoon session. Chairman: Ralph Willox

16:00–16:30: Alexander Stokes
Monodromy of monodromy manifolds and symmetries of Sakai surfaces

16:30–17:00: Ben Mitchell
Special Function Solutions of the Fifth Painlevé Equation

17:00–17:30: Tomas Lasic Latimer
Orthogonal polynomials, q-difference equations and discrete Painlevé

17:30–19:00: *Free time*

Special session: Commemoration of Prof. Masatoshi Noumi.

19:00–20:00: Speeches to commemorate Prof. Masatoshi Noumi given by colleagues and friends.

Thursday 19 June 2025

First morning session. Chairman: Danilo Latini

9:00–9:30: Orlando Ragnisco
The Volterra Maximally Superintegrable cases: novel analytical and numerical results
9:30–10:00: Andreas Vollmer
A geometric framework for second-order maximally superintegrable Hamiltonian systems
10:00–10:30: Yao-Zhong Zhang
Yang-Baxter integrability, R-matrices and exactly solvable lattice models
10:30–11:00: *Coffee break*
Second morning session. Chairman: Luc Vinet
11:00–11:30: Sikarin Yoo-Kong
The q-deformed Calogero's Goldfish model
11:30–12:00: Harry Braden
Monopoles and Affine Toda Equations
12:00–12:30: Anton Shchechkin
Quantum Painlevé/refined gauge theory correspondence
12:30–14:30: *Lunch break*
Excursion: Guided tour of Sirmione and the Grottoes of Catullo.
14:30: Meeting in front of Palazzo dei Congressi.
18:00: Return to Palazzo dei Congressi.
18:00–20:00: *Free time*
20:00–22:00: *Social dinner*

Friday 20 June 2025

First morning session. Chairman: Da-jun Zhang
9:00–9:30: Adam Doliwa
Quantum walks and the Ablowitz-Ladik spectral problem
9:30–10:00: Ievgen Makedonskyi
Duality theorems for staircase matrices
10:00–10:30: Thomas Kecker *(online)*
Geometric approach for quasi-Painlevé Hamiltonian systems
10:30–11:00: *Coffee break*
Second morning session. Chairman: Wolfgang Schief
11:00–11:30: Vladimir S. Gerdjikov *Generalized Fourier transforms for the hierarchies of MKdV equations related to $D_4^{(1)}$, $D_4^{(2)}$ and $D_4^{(3)}$ Kac-Moody algebras*
11:30–12:00: Sandra Carillo
Fifth order soliton equations: old & new results via Bäcklund tranformations
12:00–12:30: Simonetta Abenda
KP Solitons: Tropical Curves meet Grassmannians
12:30–14:00: *Lunch break*

First afternoon session. Chairman: Claude-Michel Viallet

14:00–14:30: John A. G. Roberts

Arithmetic and geometric aspects of the (symbolic) dynamics of piecewise-linear maps

14:30–15:00: Ralph Willox

Dynamical degrees for non-confining and higher order mappings from deautonomisation by singularity confinement

15:00–15:30: Takafumi Mase

Exact calculation of degrees for lattice equations: a singularity approach

15:30–16:00: *Coffee break*

Second afternoon session. Chairman: Frank Nijhoff

16:00–16:30: Tomoyuki Takenawa

An extension to the singular pattern approach for calculating the degree growth in N-dimensional birational dynamical systems

16:30–17:00: Matteo Casati

Multi-component Hamiltonian difference operators

17:00–17:30: Mats Vermeeren

Discrete Lagrangian multiforms for ABS equations revisited

17:30–18:00: *Closing remarks*

Saturday 21 June 2025

Departure day.